Section 9: Z-Transform and Linear Differential Equations

MATH241 Lecture Notes, notes pages 73–83

Notation. Scalars are plain italic (a, α, z), vectors are bold $(\boldsymbol{u}, \boldsymbol{x})$, and matrices are italic capitals $(A, B$; zero matrix O, identity $I)$. The handwritten notes underline vectors once and matrices twice.

These notes are built on top of Orhan Arikan's linear algebra course notes.

notes page 73

The Z-Transform

The z-transform is a tool for analyzing and designing discrete-time systems. It is also very useful for solving difference equations, because it turns the difference equation into an algebraic equation.

Def'n Given a sequence $x[n]$, its *z-transform* $\overline{X}(z)$ is the sum

The z-transform

$$\overline{X}(z) = \sum_{n=0}^{\infty} x[n] z^{-n},$$

defined for those values of z for which the sum converges. These values of z form the *region of convergence*, or ROC.

Two useful facts about convergence:

(i) For a finite-length sequence, the z-transform converges for every z, except possibly $z = 0$.

(ii) If the z-transform of a sequence converges at all, it converges for every $|z| > r_0$, where r_0 depends on the sequence.

Properties:

(i) *Linearity.* Suppose $x_1[n] \overset{\mathcal{Z}}{\longleftrightarrow} \overline{X}_1(z)$ with region of convergence ROC_1, and $x_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} \overline{X}_2(z)$ with region of convergence ROC_2. Then, for any constants α_1 and α_2,

$$\alpha_1 x_1[n] + \alpha_2 x_2[n] \longleftrightarrow \alpha_1 \overline{X}_1(z) + \alpha_2 \overline{X}_2(z),$$

and the region of convergence contains the overlap, $\mathrm{ROC} \supset \mathrm{ROC}_1 \cap \mathrm{ROC}_2$.

proof Split the sum into two sums:

$$\begin{aligned}
\sum_{n=0}^{\infty} (\alpha_1 x_1[n] + \alpha_2 x_2[n]) z^{-n} &= \sum_{n=0}^{\infty} (\alpha_1 x_1[n] z^{-n} + \alpha_2 x_2[n] z^{-n}) \\
&= \alpha_1 \underbrace{\sum_{n=0}^{\infty} x_1[n] z^{-n}}_{\overline{X}_1(z)} + \alpha_2 \underbrace{\sum_{n=0}^{\infty} x_2[n] z^{-n}}_{\overline{X}_2(z)} \\
&= \alpha_1 \overline{X}_1(z) + \alpha_2 \overline{X}_2(z). \quad \blacksquare
\end{aligned}$$

notes page 74

(ii) *Delay by one step.* If $x[n] \overset{\mathcal{Z}}{\longleftrightarrow} \overline{X}(z)$, then

$$x[n-1] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-1}\overline{X}(z) + x[-1].$$

The extra term $x[-1]$ is how the initial condition enters. proof Substitute $n' = n-1$ and pull one factor z^{-1} out of the sum:

$$\begin{aligned}
\sum_{n=0}^{\infty} x[\underbrace{n-1}_{n'}]z^{-n} &= \sum_{n'=-1}^{\infty} x[n']z^{-(n'+1)} \\
&= \sum_{n'=0}^{\infty} x[n']z^{-1}z^{-n'} + x[-1] \\
&= z^{-1}\underbrace{\sum_{n'=0}^{\infty} x[n']z^{-n'}}_{\overline{X}(z)} + x[-1] \\
&= z^{-1}\overline{X}(z) + x[-1]. \quad \blacksquare
\end{aligned}$$

(iii) *Delay by k steps.* If $x[n] \overset{\mathcal{Z}}{\longleftrightarrow} \overline{X}(z)$, then for $k \geq 1$

$$x[n-k] \longleftrightarrow z^{-k}\overline{X}(z) + z^{-k+1}x[-1] + z^{-k+2}x[-2] + \cdots + x[-k].$$

proof The same substitution $n' = n-k$ gives

$$\begin{aligned}
\sum_{n=0}^{\infty} x[\underbrace{n-k}_{n'}]z^{-n} &= \sum_{n'=-k}^{\infty} x[n']z^{-(n'+k)} \\
&= z^{-k}\sum_{n'=0}^{\infty} x[n']z^{-n'} + z^{-k}(x[-1]z + x[-2]z^2 + \cdots + x[-k]z^k) \\
&= z^{-k}\overline{X}(z) + z^{-k+1}x[-1] + \cdots + x[-k].
\end{aligned}$$

(iv) *The exponential sequence.* For $x[n] = a^n$, $n \geq 0$, the sum is a geometric series, which converges when $|az^{-1}| < 1$:

$$\overline{X}(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty}(az^{-1})^n = \frac{1}{1-az^{-1}}, \quad |az^{-1}| < 1 \Rightarrow \underbrace{|a| < |z|}_{\text{ROC}}.$$

(v) *The sequence na^n.* For $x[n] = na^n$, $n \geq 0$, differentiate the geometric series with respect to z:

$$\begin{aligned}
\overline{X}(z) &= \sum_{n=0}^{\infty} na^n z^{-n} = \left[\sum_{n=0}^{\infty} na^n z^{-(n+1)}\right] z = -\left(\frac{d}{dz}\sum_{n=0}^{\infty} a^n z^{-n}\right) z \\
&= -\left[\frac{d}{dz}\frac{1}{1-az^{-1}}\right] z \\
\overline{X}(z) &= \frac{az^{-1}}{(1-az^{-1})^2}. \quad \blacksquare
\end{aligned}$$

notes page 75

We can now solve difference equations by transforming them. The method is: apply the z-transform to both sides (the delay rules bring in the initial conditions), solve for $\overline{X}(z)$ algebraically, expand it in partial fractions, and invert each term.

ex/ Solve $x[n] = \frac{5}{6}x[n-1] - \frac{1}{6}x[n-2]$ with $x[-1] = -4$ and $x[-2] = -14$.

Transform both sides to the z-domain, using properties (ii) and (iii):

$$\begin{aligned}\overline{X}(z) &= \frac{5}{6}(z^{-1}\overline{X}(z) + x[-1]) - \frac{1}{6}(z^{-2}\overline{X}(z) + z^{-1}x[-1] + x[-2])\\ \overline{X}(z)\left(1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}\right) &= -\frac{20}{6} + \frac{4}{6}z^{-1} + \frac{14}{6}\\ &= \frac{4}{6}z^{-1} - 1.\end{aligned}$$

Solving for $\overline{X}(z)$ and factoring the denominator, we write it as a sum of simple fractions:

$$\begin{aligned}\overline{X}(z) &= \frac{\frac{4}{6}z^{-1} - 1}{1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}} = \frac{\frac{4}{6}z^{-1} - 1}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{3}z^{-1})}\\ &= \underbrace{\frac{A}{1 - \frac{1}{2}z^{-1}} + \frac{B}{1 - \frac{1}{3}z^{-1}}}_{\text{partial fraction expansion}}.\end{aligned}$$

To find A, multiply by $(1 - \frac{1}{2}z^{-1})$ and set $z = \frac{1}{2}$; to find B, multiply by $(1 - \frac{1}{3}z^{-1})$ and set $z = \frac{1}{3}$:

$$A = (1 - \tfrac{1}{2}z^{-1})\overline{X}(z)\big|_{z=1/2} = \frac{\frac{4}{6}\cdot 2 - 1}{1 - \frac{2}{3}} = \frac{1/3}{1/3} = 1$$

$$B = (1 - \tfrac{1}{3}z^{-1})\overline{X}(z)\big|_{z=1/3} = \frac{\frac{4}{6}\cdot 3 - 1}{1 - \frac{1}{2}\cdot 3} = \frac{1}{-1/2} = -2.$$

Each fraction is of the form in property (iv), so we can invert it:

$$\Rightarrow \overline{X}(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} - \frac{2}{1 - \frac{1}{3}z^{-1}} \overset{\mathcal{Z}^{-1}}{\longleftrightarrow} x[n] = \left(\frac{1}{2}\right)^n - 2\left(\frac{1}{3}\right)^n, \quad n \geq 0.$$

ex/ Solve $x[n] = x[n-1] - \frac{1}{4}x[n-2]$ with $x[-1] = 1$ and $x[-2] = -1$.

Transform both sides to the z-domain:

$$\begin{aligned}\overline{X}(z) &= (z^{-1}\overline{X}(z) + x[-1]) - \frac{1}{4}(z^{-2}\overline{X}(z) + z^{-1}x[-1] + x[-2])\\ &= z^{-1}\overline{X}(z) + 1 - \frac{1}{4}z^{-2}\overline{X}(z) - \frac{1}{4}z^{-1} + \frac{1}{4}.\\ \Rightarrow \overline{X}(z)\left(1 - z^{-1} + \frac{1}{4}z^{-2}\right) &= \frac{5}{4} - \frac{1}{4}z^{-1}.\end{aligned}$$

This time the denominator is a perfect square, $(1-\frac{1}{2}z^{-1})^2$, so the partial fraction expansion contains a repeated factor:

$$\Rightarrow \overline{X}(z) = \frac{\frac{5}{4} - \frac{1}{4}z^{-1}}{1 - z^{-1} + \frac{1}{4}z^{-2}} = \frac{\frac{5}{4} - \frac{1}{4}z^{-1}}{(1 - \frac{1}{2}z^{-1})^2} = \frac{A}{1 - \frac{1}{2}z^{-1}} + \frac{B}{(1 - \frac{1}{2}z^{-1})^2}.$$

notes page 76

Clearing denominators and comparing the constant terms and the coefficients of z^{-1}:

$$\Rightarrow \frac{5}{4} - \frac{1}{4}z^{-1} = A(1 - \tfrac{1}{2}z^{-1}) + B = A + B - \frac{A}{2}z^{-1}.$$

$$\Rightarrow \begin{array}{c} A + B = 5/4 \\ -A/2 = -1/4 \Rightarrow A = 1/2 \Rightarrow B = 5/4 - A = 5/4 - 1/2 = 3/4. \end{array}$$

Hence

$$\overline{X}(z) = \frac{1}{2}\frac{1}{1-\frac{1}{2}z^{-1}} + \frac{3}{4}\frac{1}{(1-\frac{1}{2}z^{-1})^2}.$$

To invert these we use two transform pairs. The first is property (iv). The second follows by multiplying the geometric series $\sum_n w^n = \frac{1}{1-w}$ by itself with $w = az^{-1}$, since $(\sum_n w^n)^2 = \sum_n (n+1)w^n$:

$$\frac{1}{1-az^{-1}} \overset{\mathcal{Z}^{-1}}{\longleftrightarrow} a^n, \qquad \frac{1}{(1-az^{-1})^2} \overset{\mathcal{Z}^{-1}}{\longleftrightarrow} (n+1)a^n, \qquad n \geq 0.$$

Therefore

$$\overline{X}(z) \overset{\mathcal{Z}^{-1}}{\longleftrightarrow} x[n] = \frac{1}{2}\left(\frac{1}{2}\right)^n + \frac{3}{4}(n+1)\left(\frac{1}{2}\right)^n, \quad n \geq 0,$$

which simplifies to

Solution

$$x[n] = \frac{5}{4}\left(\frac{1}{2}\right)^n + \frac{3}{4}n\left(\frac{1}{2}\right)^n, \quad n \geq 0.$$

Linear Constant Coefficient Differential Equations

(Chapter 3, sections 4 and 5 in the textbook.)

Differential equations describe continuous-time systems in the way that difference equations describe discrete-time ones. A homogeneous linear differential equation with constant coefficients of order n has the form

$$\underbrace{\frac{d^n y}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y = 0}_{\text{homogeneous } n^{\text{th}}\text{-order linear constant coefficient DE}}$$

The simplest case is first order: $\frac{dy}{dx} + a_1 y = 0$. Try a solution of the form $y(x) = e^{\lambda x}$. Then $\frac{dy}{dx} = \lambda e^{\lambda x}$, so

$$\lambda e^{\lambda x} + a_1 e^{\lambda x} = 0 \Rightarrow \lambda = -a_1.$$

Hence the general solution is $Ce^{-a_1 x}$. To determine the constant C we need an initial condition such as $y(0)$:

$$y(0) = Ce^{-a_1 \cdot 0} = C \;\Rightarrow\; y(x) = y(0)e^{-a_1 x}.$$

— notes page 77 —

ex/ A second-order equation. Consider

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = 0 \qquad \text{which is also written } y'' + a_1 y' + a_2 y = 0.$$

We again try solutions of the form $y(x) = e^{\lambda x}$. Each derivative just brings down another factor λ:

$$\Rightarrow \lambda^2 e^{\lambda x} + a_1\lambda e^{\lambda x} + a_2 e^{\lambda x} = 0$$

$$\Rightarrow (\lambda^2 + a_1\lambda + a_2)e^{\lambda x} = 0 \Rightarrow \underbrace{\lambda^2 + a_1\lambda + a_2 = 0}_{\text{characteristic equation}}.$$

Since $e^{\lambda x}$ is never zero, λ has to be a root of the characteristic equation:

$$\lambda_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2}, \quad \lambda_2 = \frac{-a_1 - \sqrt{a_1^2 - 4a_2}}{2}.$$

For real a_1 and a_2 these are either two real roots or a pair of complex conjugate roots. The general solution is a combination of the two:

General solution, distinct roots

$$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}.$$

Given initial conditions $y(0)$ and $y'(0)$, the constants C_1 and C_2 are determined uniquely.

ex/ Real roots. Solve $y'' - 9y = 0$ with $y(0) = 4$ and $y'(0) = -5$. The characteristic equation $\lambda^2 - 9 = 0$ gives $\lambda_1 = 3$ and $\lambda_2 = -3$, so

$$y(x) = C_1 e^{3x} + C_2 e^{-3x}.$$

The initial conditions give two equations for C_1 and C_2:

$$\left.\begin{aligned} 4 &= C_1 + C_2 \\ -5 &= 3C_1 - 3C_2 \end{aligned}\right\} \Rightarrow 6C_1 = 7 \Rightarrow C_1 = \frac{7}{6}, \qquad C_2 = 4 - C_1 = 4 - \frac{7}{6} = \frac{17}{6}.$$

$$\Rightarrow y(x) = \frac{7}{6}e^{3x} + \frac{17}{6}e^{-3x}.$$

ex/ Imaginary roots. Solve $y'' + 9y = 0$. The characteristic equation $\lambda^2 + 9 = 0$ gives $\lambda_1 = j3$ and $\lambda_2 = -j3$, so $y(x) = C_1 e^{j3x} + C_2 e^{-j3x}$. Complex exponentials are not convenient to read, so we rewrite the solution with sines and cosines, using Euler's formula in the form $\cos 3x = \frac{1}{2}(e^{j3x} + e^{-j3x})$ and $\sin 3x = \frac{1}{2j}(e^{j3x} - e^{-j3x})$:

$$\begin{aligned} \Rightarrow y(x) &= C_1 e^{j3x} + C_2 e^{-j3x} \\ &= \left(\frac{C_1 + C_2}{2} + \frac{C_1 - C_2}{2}\right) e^{j3x} + \left(\frac{C_2 + C_1}{2} + \frac{C_2 - C_1}{2}\right) e^{-j3x} \\ \Rightarrow y(x) &= \left(\frac{C_1 + C_2}{2}\right)(e^{j3x} + e^{-j3x}) + j\left(\frac{C_1 - C_2}{2j}\right)(e^{j3x} - e^{-j3x}) \\ &= (C_1 + C_2)\cos 3x + j(C_1 - C_2)\sin 3x \\ &= A\cos 3x + B\sin 3x \qquad (A \text{ and } B \text{ can be real or complex}). \end{aligned}$$

notes page 78

ex/ Complex roots with a real part. Solve $y'' + 4y' + 7y = 0$ with $y(0) = 1$ and $y'(0) = 0$. The characteristic equation is $\lambda^2 + 4\lambda + 7 = 0$, so

$$\Rightarrow \lambda = \frac{-4 \pm \sqrt{16 - 28}}{2} = \frac{-4 \pm \sqrt{-12}}{2} = -2 \pm j\sqrt{3}.$$

The real part -2 gives a decaying factor e^{-2x}, and the imaginary part $\pm\sqrt{3}$ gives oscillation:

$$\begin{aligned} \Rightarrow y(x) &= C_1 e^{(-2+j\sqrt{3})x} + C_2 e^{(-2-j\sqrt{3})x} \\ &= e^{-2x}(C_1 e^{j\sqrt{3}x} + C_2 e^{-j\sqrt{3}x}) \\ &= e^{-2x}(A\cos\sqrt{3}x + B\sin\sqrt{3}x). \end{aligned}$$

Now apply the initial conditions. First, $y(0) = 1$ gives $A = 1$. Differentiating y and evaluating at $x = 0$,

$$y'(0) = \left[-2e^{-2x}(A\cos\sqrt{3}x + B\sin\sqrt{3}x) + e^{-2x}(-\sqrt{3}A\sin\sqrt{3}x + \sqrt{3}B\cos\sqrt{3}x)\right]\Big|_{x=0} = -2A + \sqrt{3}B = 0,$$

so $B = \dfrac{2}{\sqrt{3}}$ and

$$\Rightarrow y(x) = e^{-2x}\left(\cos\sqrt{3}x + \frac{2}{\sqrt{3}}\sin\sqrt{3}x\right).$$

Theorem 3.4.1

Combining solutions as above only works if the exponentials $e^{\lambda_i x}$ really are independent of each other. This is guaranteed when the roots are distinct.

Theorem 3.4.1
If the characteristic equation of an n^{th}-order linear constant coefficient DE has n distinct roots $\lambda_1, \ldots, \lambda_n$, then the set $\{e^{\lambda_1 x}, \ldots, e^{\lambda_n x}\}$ is linearly independent on any interval $I \subset \mathbb{R}$.

proof Suppose, for contradiction, that they are linearly dependent. Then there are coefficients, not all zero, with

$$\alpha_1 e^{\lambda_1 x} + \alpha_2 e^{\lambda_2 x} + \cdots + \alpha_n e^{\lambda_n x} = 0, \quad \forall x \in I.$$

Differentiating this identity repeatedly, each derivative multiplies the i^{th} term by another λ_i:

$$\alpha_1 \lambda_1 e^{\lambda_1 x} + \alpha_2 \lambda_2 e^{\lambda_2 x} + \cdots + \alpha_n \lambda_n e^{\lambda_n x} = 0$$

$$\vdots$$

$$\alpha_1 \lambda_1^{n-1} e^{\lambda_1 x} + \alpha_2 \lambda_2^{n-1} e^{\lambda_2 x} + \cdots + \alpha_n \lambda_n^{n-1} e^{\lambda_n x} = 0.$$

These n equations can be written as one matrix equation:

$$\Rightarrow \underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix}}_{\text{a Vandermonde matrix, invertible because the } \lambda_i \text{ are distinct}} \begin{bmatrix} \alpha_1 e^{\lambda_1 x} \\ \alpha_2 e^{\lambda_2 x} \\ \vdots \\ \alpha_n e^{\lambda_n x} \end{bmatrix} = \mathbf{0}$$

$$\Rightarrow \begin{bmatrix} \alpha_1 e^{\lambda_1 x} \\ \vdots \\ \alpha_n e^{\lambda_n x} \end{bmatrix} = \mathbf{0} \Rightarrow \alpha_i e^{\lambda_i x} = 0 \Rightarrow \alpha_i = 0.$$

All the coefficients are zero, which contradicts the assumption. ■

— notes page 79 —

Theorem 3.4.2 — Repeated Roots Case

If a root is repeated, we have fewer than n exponentials, so we need more solutions. The missing ones are obtained by multiplying by powers of x.

Theorem 3.4.2
If λ_1 is a root of order k of the characteristic equation, then $e^{\lambda_1 x}, xe^{\lambda_1 x}, \ldots, x^{k-1}e^{\lambda_1 x}$ are linearly independent solutions of the differential equation.

proof Write the equation in operator form as $L[y] = 0$, where

$$L = \frac{d^n}{dx^n} + a_1 \frac{d^{n-1}}{dx^{n-1}} + \cdots + a_{n-1}\frac{d}{dx} + a_n.$$

Applying L to an exponential just multiplies it by the characteristic polynomial. Because λ_1 is a root of multiplicity k, the polynomial factors, with a leftover polynomial $p(\lambda)$ of degree $n-k$:

$$\begin{aligned} L[e^{\lambda x}] &= (\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n)e^{\lambda x} \\ &= (\lambda - \lambda_1)^k p(\lambda) e^{\lambda x}. \end{aligned}$$

At $\lambda = \lambda_1$ this is zero, so $L[e^{\lambda_1 x}] = 0$ and $e^{\lambda_1 x}$ is a solution.
To show that $xe^{\lambda_1 x}$ is also a solution, differentiate $L[e^{\lambda x}]$ with respect to λ:

$$\begin{aligned}\frac{d}{d\lambda}L[e^{\lambda x}] &= \frac{d}{d\lambda}[(\lambda-\lambda_1)^k p(\lambda)e^{\lambda x}] \\ &= k(\lambda-\lambda_1)^{k-1}p(\lambda)e^{\lambda x} + (\lambda-\lambda_1)^k \frac{d}{d\lambda}(p(\lambda)e^{\lambda x}).\end{aligned}$$

On the other hand, the derivative with respect to λ can be moved inside L, which acts on x:

$$\frac{d}{d\lambda}L[e^{\lambda x}] = L\left[\frac{d}{d\lambda}e^{\lambda x}\right] = L[xe^{\lambda x}].$$

Evaluate both expressions at $\lambda = \lambda_1$. Both terms of the right-hand side contain a positive power of $(\lambda - \lambda_1)$ (this needs $k \geq 2$), so they vanish:

$$L[xe^{\lambda x}]\Big|_{\lambda=\lambda_1} = \underbrace{\left[k(\lambda-\lambda_1)^{k-1}p(\lambda)e^{\lambda x} + (\lambda-\lambda_1)^k \frac{d}{d\lambda}(p(\lambda)e^{\lambda x})\right]}_{0}\Bigg|_{\lambda=\lambda_1}.$$

$$\Rightarrow L[xe^{\lambda_1 x}] = 0 \Rightarrow xe^{\lambda_1 x} \text{ is also a solution.}$$

Differentiating more times with respect to λ in the same way shows that $x^2e^{\lambda_1 x}, \ldots, x^{k-1}e^{\lambda_1 x}$ are solutions too. ■

example Solve $y^{(iv)} - 8y''' + 26y'' - 40y' + 25y = 0$. The characteristic equation is $\lambda^4 - 8\lambda^3 + 26\lambda^2 - 40\lambda + 25 = 0$. It has repeated complex roots: $\lambda_1 = 2 + j$ with multiplicity 2 and $\lambda_2 = 2 - j$ with multiplicity 2. Each root contributes its exponential and x times its exponential, so the general solution is

$$\begin{aligned}y(x) &= (C_1 + C_2x)e^{(2+j)x} + (C_3 + C_4x)e^{(2-j)x} \\ &= e^{2x}(C_1e^{jx} + C_3e^{-jx} + x(C_2e^{jx} + C_4e^{-jx})) \\ &= e^{2x}((A\cos x + B\sin x) + x(C\cos x + D\sin x)).\end{aligned}$$

—— notes page 80 ——

Def'n — Stability: consider the homogeneous differential equation

$$\frac{d^n y}{dt^n} + a_1\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_{n-1}\frac{dy}{dt} + a_n y = 0 \qquad (\mathrm{DE}_n)$$

with general solution $y(t) = C_1y_1(t) + \cdots + C_ny_n(t)$. It is called *stable* if all of its solutions stay bounded for $t > 0$, that is,

$$|y_i(t)| < M_i < \infty, \quad 1 \leq i \leq n, \quad t > 0.$$

Theorem 3.4.3

Stability is decided by where the roots of the characteristic equation lie in the complex plane.

Theorem 3.4.3
(DE_n) is stable if and only if its characteristic equation has no roots to the right of the imaginary axis, and any roots on the imaginary axis are non-repeated.

proof (i) *Non-repeated roots.* A non-repeated root $\lambda = a + jb$ contributes a solution $e^{\lambda t} = e^{at}e^{jbt}$. If $a \leq 0$ this stays bounded:

$$|e^{\lambda t}| = |e^{at}e^{jbt}| = |e^{at}|\underbrace{|e^{jbt}|}_{1} = e^{at} \leq 1, \quad t > 0.$$

(ii) *Repeated roots.* A repeated root $\lambda = a + jb$ of order k contributes the k solutions $e^{\lambda t}, te^{\lambda t}, \ldots, t^{k-1}e^{\lambda t}$. For $a < 0$ we show that all of them are bounded. For $t > 0$,

$$|t^i e^{\lambda t}| = |t^i| \cdot |e^{at} \cdot e^{jbt}| = t^i e^{at}.$$

Since $t^i e^{at} \to 0$ as $t \to \infty$ when $a < 0$, and $t^i e^{at}$ is finite on every bounded interval $0 < t < T$, the function $t^i e^{at}$ is bounded for $t > 0$.

(iii) *Instability.* If a simple root has $a > 0$, or a repeated root has $a \geq 0$, the solutions it contributes grow without bound, so the system is unstable. ■

Theorem 3.4.4 — Coefficients of mixed sign

There is a quick necessary check for stability that needs no root finding: look at the signs of the coefficients.

Theorem 3.4.4
If the coefficients of the characteristic equation are real and of mixed sign (at least one positive and at least one negative), then there is at least one root λ with $\Re\{\lambda\} > 0$, so the system is unstable.

—— notes page 81 ——

Note that the theorem does not say that every unstable system has coefficients of mixed sign. It says only that mixed signs are enough to guarantee instability. (An example appears below.)

Proof We prove the contrapositive: if a system is stable, its characteristic polynomial cannot have coefficients of mixed sign. A stable system has all its roots in the closed left half of the complex plane, $\Re\{\lambda\} \leq 0$. Since the coefficients are real, the complex roots come in conjugate pairs, so the monic characteristic polynomial factors into two kinds of factors:

(i) a factor $(\lambda - \lambda_r) = \lambda + c$ for each real root $\lambda_r = -c \leq 0$, which has non-negative coefficients 1 and c;

(ii) a factor for each conjugate pair $a \pm jb$ with $a \leq 0$,

$$(\lambda - (a + jb))(\lambda - (a - jb)) = \lambda^2 - 2a\lambda + (a^2 + b^2),$$

which also has non-negative coefficients, because $-2a \geq 0$ and $a^2 + b^2 \geq 0$.

When two polynomials with non-negative coefficients are multiplied, every coefficient of the product is a sum of products of non-negative numbers, so it is non-negative too. Hence the characteristic polynomial of a stable system has only non-negative coefficients, and in particular its coefficients cannot be of mixed sign. ■

—— notes page 82 ——

Conversely, a polynomial of mixed sign must have a root with a positive real part, which makes the system unstable. This is the statement of the theorem.

Now we see why the converse of the theorem fails: unstable roots do not always produce mixed signs. For an unstable real root $\lambda_r > 0$, multiplying a polynomial $q(\lambda)$ with non-negative coefficients

by $(\lambda - \lambda_r)$ gives a polynomial whose last non-zero coefficient is negative, so the signs do mix. But for an unstable pair of complex roots, $a > 0$, the factor $\lambda^2 - 2a\lambda + (a^2 + b^2)$ itself has mixed signs, and multiplying it by another polynomial may or may not leave the result with mixed signs.

Note that even if the added roots are unstable, $p(\lambda)$ may have no sign mix. For example, take

$$q(\lambda) = (\lambda + 1), \quad \lambda^2 - 2a\lambda + (a^2 + b^2) = \lambda^2 - \epsilon\lambda + 1, \quad \epsilon > 0,$$

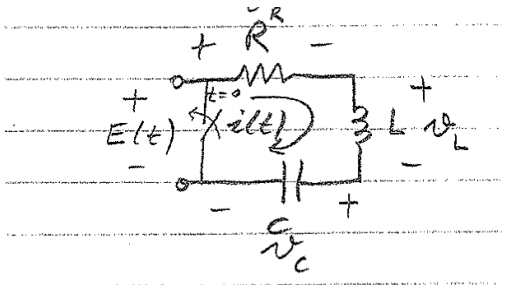
which has the unstable pair $a = \epsilon/2 > 0$. Then

$$\begin{aligned} p(\lambda) &= q(\lambda) \cdot (\lambda^2 - 2a\lambda + (a^2 + b^2)) = (\lambda + 1)(\lambda^2 - \epsilon\lambda + 1) \\ &= \lambda^3 + (1 - \epsilon)\lambda^2 + (1 - \epsilon)\lambda + 1. \end{aligned}$$

If $0 < \epsilon < 1$, all the coefficients of $p(\lambda)$ are positive, so the coefficients do not have mixed signs, even though the system is unstable.

Application Example: Oscillator Circuit

Here is a physical system described by a second-order differential equation: a resistor R, an inductor L and a capacitor C connected in a loop to a voltage source $E(t)$.



By Kirchhoff's voltage law, the source voltage equals the sum of the voltages across the three elements. Each element has its own voltage–current law:

$$\begin{aligned} E(t) &= v_R(t) + v_L(t) + v_C(t) \\ v_R(t) &= Ri(t) \\ v_L(t) &= L\frac{di(t)}{dt} \\ i(t) &= C\frac{dv_C(t)}{dt} \Rightarrow v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)\,d\tau. \end{aligned}$$

Substituting these laws into the first one, and then differentiating once to remove the integral, gives a second-order differential equation for the current:

$$\begin{aligned} &\Rightarrow E(t) = Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int_{-\infty}^{t} i(\tau)\,d\tau \\ &\Rightarrow E'(t) = L\frac{d^2i(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{1}{C}i(t). \end{aligned}$$

—— notes page 83 ——

Consider the free oscillator, with no source, $E(t) = 0$. Dividing by L,

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = 0, \qquad \text{characteristic equation: } \lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0.$$

$$\Rightarrow \lambda_1 = \frac{-R/L + \sqrt{R^2/L^2 - 4/LC}}{2}, \quad \lambda_2 = \frac{-R/L - \sqrt{R^2/L^2 - 4/LC}}{2}.$$

The sign of the number under the square root decides the behavior, and it depends on the resistance R.

(i) *Large resistance.* If $\dfrac{R^2}{L^2} - \dfrac{4}{LC} > 0$, which is equivalent to $R^2C - 4L > 0$, that is,

$$R > 2\sqrt{\frac{L}{C}},$$

then there are two real roots, both negative (the square root is smaller than R/L). The current is

$$i(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}, \quad \lambda_1 < 0, \quad \lambda_2 < 0.$$

There is no oscillation: this is the *overdamped* case.

(ii) *The borderline.* If $R = 2\sqrt{L/C}$, then $\lambda_1 = \lambda_2 = -\dfrac{R}{2L}$ and

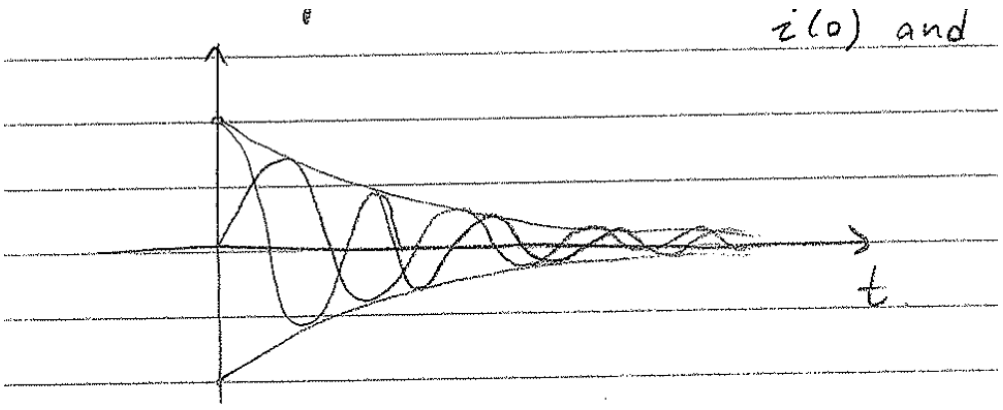
$$i(t) = C_1 e^{\lambda_1 t} + C_2 t e^{\lambda_1 t}.$$

There is again no oscillation: this is the *critically damped* case.

(iii) *Small resistance.* If $R < 2\sqrt{L/C}$, the roots are a complex conjugate pair,

$$\lambda_1 = \frac{-R/L + \sqrt{R^2/L^2 - 4/LC}}{2} = \frac{-R/L + j\sqrt{4/LC - R^2/L^2}}{2} = \lambda_2^*.$$

The real part makes the oscillation decay, and the imaginary part sets its frequency:

$$i(t) = \underbrace{e^{-\frac{R}{2L}t}}_{\text{decay envelope (time constant } 2L/R)} (A\cos(\omega t) + B\sin(\omega t)), \qquad \underbrace{\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}}_{\text{damped natural frequency}}.$$

These are *underdamped* oscillations. The constants A and B are determined from $i(0)$ and $i'(0)$.