

Section 11: Systems of Linear Differential Equations

MATH241 Lecture Notes, notes pages 89–96 + extra notes

Notation. Scalars are plain italic (a, α, z), vectors are bold ($\mathbf{u}, \mathbf{x}$), and matrices are italic capitals (A, B ; zero matrix O , identity I). The handwritten notes underline vectors once and matrices twice.

These notes are built on top of Orhan Arikan's linear algebra course notes.

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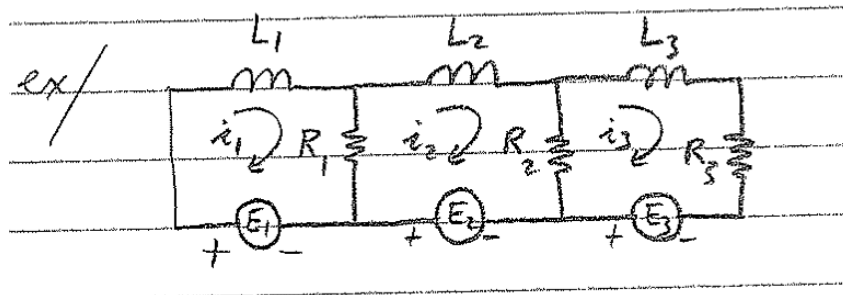
Systems of Linear Differential Equations

Many systems have several unknown functions that influence each other. Their equations form a *system of linear differential equations*:

$$\left. \begin{aligned} a_{11}(t)x'_1 + \cdots + a_{1n}(t)x'_n + b_{11}(t)x_1 + \cdots + b_{1n}(t)x_n &= f_1(t) \\ &\vdots \\ a_{n1}(t)x'_1 + \cdots + a_{nn}(t)x'_n + b_{n1}(t)x_1 + \cdots + b_{nn}(t)x_n &= f_n(t) \end{aligned} \right\}.$$

The $a_{ij}(t)$ and $b_{ij}(t)$ are known coefficients, and the $f_i(t)$ are given *forcing functions* (inputs).

ex/ An electrical circuit with three loops gives such a system. Applying Kirchhoff's voltage law to each loop gives three equations for the unknown currents i_1, i_2, i_3 :



$$\begin{aligned} L_1 i'_1 + R_1(i_1 - i_2) &= E_1(t) \\ L_2 i'_2 + R_2(i_2 - i_3) + R_1(i_2 - i_1) &= E_2(t) \\ L_3 i'_3 + R_3 i_3 + R_2(i_3 - i_2) &= E_3(t) \end{aligned}$$

Collecting the terms in each current gives the standard layout:

$$\begin{aligned} L_1 i'_1 + R_1 i_1 - R_1 i_2 &= E_1(t) \\ L_2 i'_2 - R_1 i_1 + (R_2 + R_1) i_2 - R_2 i_3 &= E_2(t) \\ L_3 i'_3 - R_2 i_2 + (R_3 + R_2) i_3 &= E_3(t). \end{aligned}$$

Theorem 3.9.1

In the standard form, each equation has one derivative on the left, and solutions exist and are unique.

Theorem 3.9.1: existence and uniqueness

Let $a_{ij}(t)$, $1 \leq i, j \leq n$, and $f_i(t)$, $1 \leq i \leq n$, be continuous on a closed interval I , and let $x_i(t_0) = b_i$, $1 \leq i \leq n$, for some $t_0 \in I$. Then the system

$$\begin{aligned} x_1' &= a_{11}(t)x_1 + \cdots + a_{1n}(t)x_n + f_1(t) \\ &\vdots \\ x_n' &= a_{n1}(t)x_1 + \cdots + a_{nn}(t)x_n + f_n(t) \end{aligned}$$

has a unique solution on the entire interval I .

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Note that

- (i) In this standard form the left-hand side contains each derivative on its own. A general system can be brought to this form by Gauss–Jordan elimination on the equations (possible when the matrix $[a_{ij}(t)]$ of derivative coefficients is invertible).
- (ii) The system can be written in vector form, which is how we will work with it from now on:

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}$$

$$\text{where } \mathbf{x} = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}, A(t) = [a_{ij}(t)]_{n \times n} \text{ and } \mathbf{f} = \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}.$$

ex/ Solve a 2×2 system by the Laplace transform. First we bring it to standard form by subtracting the first equation from the second:

$$\left. \begin{aligned} x_1' &= x_1 + x_2 + 3t \\ x_1' + x_2' &= 5x_1 + 2x_2 + 5 \end{aligned} \right\} \Rightarrow \begin{aligned} x_1' &= x_1 + x_2 + 3t \\ x_2' &= 4x_1 + x_2 + 5 - 3t. \end{aligned}$$

In vector form,

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3t \\ 5 - 3t \end{bmatrix}.$$

In the Laplace transform domain this becomes an algebraic matrix equation,

$$(s\bar{\mathbf{x}} - \mathbf{x}_0) = A\bar{\mathbf{x}} + \mathbf{F}, \quad \bar{\mathbf{x}} = \begin{bmatrix} \bar{X}_1(s) \\ \bar{X}_2(s) \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} F_1(s) \\ F_2(s) \end{bmatrix}$$

where the transforms of the forcing terms are

$$\begin{aligned} F_1(s) &= \int_0^\infty 3te^{-st} dt \\ &= \underbrace{-\frac{3te^{-st}}{s} \Big|_0^\infty}_0 + \frac{3}{s} \underbrace{\int_0^\infty e^{-st} dt}_{-\frac{1}{s}e^{-st} \Big|_0^\infty = \frac{1}{s}} = \frac{3}{s^2}, \\ F_2(s) &= \int_0^\infty (5 - 3t)e^{-st} dt = \frac{5}{s} - \frac{3}{s^2}. \end{aligned}$$

Collecting the unknowns on the left and solving,

$$\Rightarrow (sI - A)\bar{\mathbf{x}} = \mathbf{x}_0 + \mathbf{F} \quad \Rightarrow \quad \bar{\mathbf{x}} = (sI - A)^{-1}(\mathbf{x}_0 + \mathbf{F}),$$

where, for our matrix, the 2×2 inverse formula gives

$$(sI - A) = \begin{bmatrix} s-1 & -1 \\ -4 & s-1 \end{bmatrix}, \quad (sI - A)^{-1} = \frac{1}{s^2 - 2s - 3} \begin{bmatrix} s-1 & 1 \\ 4 & s-1 \end{bmatrix}.$$

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Note that $s^2 - 2s - 3 = (s-3)(s+1)$. Multiplying out gives two parts, one that depends on the initial conditions and one that depends on the forcing:

$$\begin{aligned} \Rightarrow \bar{\mathbf{x}} &= \frac{1}{(s-3)(s+1)} \begin{bmatrix} s-1 & 1 \\ 4 & s-1 \end{bmatrix} \left(\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} 3/s^2 \\ 5/s - 3/s^2 \end{bmatrix} \right) \\ &= \begin{bmatrix} \frac{(s-1)x_1(0) + x_2(0)}{(s-3)(s+1)} \\ \frac{4x_1(0) + (s-1)x_2(0)}{(s-3)(s+1)} \end{bmatrix} + \begin{bmatrix} \frac{(s-1)3}{s^2(s-3)(s+1)} + \frac{5s-3}{s^2(s-3)(s+1)} \\ \frac{12}{s^2(s-3)(s+1)} + \frac{(5s-3)(s-1)}{s^2(s-3)(s+1)} \end{bmatrix} \end{aligned}$$

Each entry is a ratio of polynomials. Expanding each in partial fractions gives

$$= \begin{bmatrix} \frac{2x_1(0) + x_2(0)}{4(s-3)} + \frac{2x_1(0) - x_2(0)}{4(s+1)} \\ \frac{x_2(0) + 2x_1(0)}{2(s-3)} + \frac{x_2(0) - 2x_1(0)}{2(s+1)} \end{bmatrix} + \begin{bmatrix} -\frac{4}{s} + \frac{2}{s^2} + \frac{1/2}{s-3} + \frac{7/2}{s+1} \\ \frac{6}{s} - \frac{5}{s^2} + \frac{1}{s-3} - \frac{7}{s+1} \end{bmatrix}.$$

Inverting term by term, using $\frac{1}{s-a} \leftrightarrow e^{at}$, $\frac{1}{s} \leftrightarrow 1$ and $\frac{1}{s^2} \leftrightarrow t$, gives the solution:

$$\begin{aligned} \mathbf{x}(t) &= \underbrace{\begin{bmatrix} \left(\frac{2x_1(0) + x_2(0)}{4} \right) e^{3t} + \left(\frac{2x_1(0) - x_2(0)}{4} \right) e^{-t} \\ \frac{x_2(0) + 2x_1(0)}{2} e^{3t} + \frac{x_2(0) - 2x_1(0)}{2} e^{-t} \end{bmatrix}}_{\text{determined by the initial conditions } x_1(0), x_2(0)} \\ &\quad + \underbrace{\begin{bmatrix} -4 + 2t + \frac{1}{2}e^{3t} + \frac{7}{2}e^{-t} \\ 6 - 5t + e^{3t} - 7e^{-t} \end{bmatrix}}_{\text{particular solution for zero initial conditions}}. \end{aligned}$$

There is another approach, based on *state-transition matrices*. We want to solve

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t) \quad \text{with initial value } \mathbf{x}(0) = \mathbf{x}_0.$$

Observations

We start with the homogeneous system $\mathbf{x}' = A(t)\mathbf{x}$.

(i) The set of solutions of the homogeneous system

$$\mathbf{x}' = A(t)\mathbf{x}$$

is an n -dimensional vector space of functions. It is closed under the vector space operations, because if $\mathbf{x}_1(t)$ is a solution then so is $\alpha\mathbf{x}_1(t)$, and if $\mathbf{x}_2(t)$ is also a solution then so is $\mathbf{x}_1(t) + \mathbf{x}_2(t)$.

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(ii) Suppose $\{\phi_1(t), \dots, \phi_n(t)\}$ are n linearly independent solutions of $\mathbf{x}' = A(t)\mathbf{x}$. Then every solution has the form

$$\mathbf{x}(t) = \alpha_1\phi_1(t) + \dots + \alpha_n\phi_n(t)$$

for some constants α_i . In matrix form, with the solutions as the columns of a *fundamental matrix* $\Phi(t)$,

$$\mathbf{x}(t) = \Phi(t)\boldsymbol{\alpha}$$

where $\Phi(t) = [\phi_1(t) \cdots \phi_n(t)]$ and $\boldsymbol{\alpha} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$.

(iii) *Notation.* We write

$$\Phi(t; t_0, X_0) = [\phi(t; t_0, \mathbf{x}_1) \cdots \phi(t; t_0, \mathbf{x}_n)],$$

where $\phi(t; t_0, \mathbf{x}_i)$ is the solution of

$$\mathbf{x}' = A\mathbf{x}, \quad \text{with } \mathbf{x}(t_0) = \mathbf{x}_i.$$

(iv) The special case $X_0 = I$, that is, starting each solution from a standard basis vector, is called the state-transition matrix:

$$\Phi(t; t_0) = [\phi(t; t_0, \mathbf{e}_1) \cdots \phi(t; t_0, \mathbf{e}_n)].$$

(v) The unique solution of $\mathbf{x}' = A(t)\mathbf{x}$ with $\mathbf{x}(t_0) = \mathbf{x}_0$ can be written with either matrix. Write the initial vector in the starting vectors $\mathbf{x}_i$, with coefficients $\boldsymbol{\alpha}$, and use the fact that a combination of solutions is a solution:

$$\phi(t; t_0, \mathbf{x}_0) = \sum_{i=1}^n \phi(t; t_0, \mathbf{x}_i) \alpha_i = \Phi(t; t_0, X_0) \boldsymbol{\alpha},$$

where

$$\mathbf{x}_0 = \alpha_1 \mathbf{x}_1 + \cdots + \alpha_n \mathbf{x}_n = \underbrace{[\mathbf{x}_1 \cdots \mathbf{x}_n]}_{X_0} \boldsymbol{\alpha} \Rightarrow \boldsymbol{\alpha} = X_0^{-1} \mathbf{x}_0.$$

So

$$\Rightarrow \phi(t; t_0, \mathbf{x}_0) = \Phi(t; t_0, X_0) X_0^{-1} \mathbf{x}_0.$$

The choice $X_0 = I$ gives the state-transition matrix:

The state-transition matrix

$$\phi(t; t_0, \mathbf{x}_0) = \Phi(t; t_0) \mathbf{x}_0 \quad \Longrightarrow \quad \Phi(t; t_0, X_0) = \Phi(t; t_0) X_0.$$

In words, the state-transition matrix carries the state at time t_0 to the state at time t .

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Properties of the State-Transition Matrix

Three properties of $\Phi(t; t_0)$ make it easy to work with. Throughout, I is the interval on which the solutions exist.

(i) *Non-singularity.* $\Phi(t; t_0, X_0)$ is non-singular for all t and t_0 in I , whenever X_0 is non-singular.

proof We argue by contradiction. Assume that $\Phi(t_1; t_0, X_0)$ is singular for some t_1 and t_0 in I . Then there is a vector $\mathbf{c} \neq \mathbf{0}$ with

$$\Phi(t_1; t_0, X_0) \mathbf{c} = \mathbf{0}.$$

Define $\boldsymbol{\psi}(t) = \Phi(t; t_0, X_0) \mathbf{c}$. This is a combination of solutions, hence a solution of $\mathbf{x}' = A(t)\mathbf{x}$, and it satisfies $\boldsymbol{\psi}(t_1) = \mathbf{0}$. The zero function is also a solution with this initial value, and the solution is unique (Theorem 3.9.1), so $\boldsymbol{\psi}(t) = \mathbf{0}$ for all t , the zero function. In particular

$$\boldsymbol{\psi}(t_0) = X_0 \mathbf{c} = \mathbf{0},$$

which says X_0 is singular, contradicting our assumption. ■

(ii) *The composition rule.* For all t, t_1, t_0 in I ,

$$\Phi(t; t_1)\Phi(t_1; t_0) = \Phi(t; t_0).$$

In words: going from t_0 to t_1 and then on to t is the same as going from t_0 to t directly.

proof Let $\Psi(t) = \Phi(t; t_1)\Phi(t_1; t_0)$. Differentiating, and using that $\Phi(t; t_1)$ solves the matrix equation,

$$\begin{aligned}\Psi' &= \Phi'(t; t_1)\Phi(t_1; t_0) = A(t) \underbrace{\Phi(t; t_1)\Phi(t_1; t_0)}_{\Psi(t)} \\ &\Rightarrow \Psi' = A(t)\Psi(t),\end{aligned}$$

and at $t = t_1$ the initial value is

$$\Psi(t_1) = \underbrace{\Phi(t_1; t_1)}_I \Phi(t_1; t_0) = \Phi(t_1; t_0).$$

Thus $\Psi(t)$ is the unique solution of $X' = A(t)X$ with $X(t_1) = \Phi(t_1; t_0)$. But $\Phi(t; t_0)$ solves the same problem, so by uniqueness

$$\Psi(t) = \Phi(t; t_0) \Rightarrow \Phi(t; t_0) = \Phi(t; t_1)\Phi(t_1; t_0). \quad \blacksquare$$

(iii) *The inverse.* Taking $t = t_0$ in rule (ii) gives

$$\Phi(t_0; t_0) = I = \Phi(t_0; t_1)\Phi(t_1; t_0) \quad \forall t_0, t_1 \in I,$$

so $\Phi(t_0; t_1)$ is the inverse of $\Phi(t_1; t_0)$. Renaming the variables:

Inverse of the state-transition matrix

$$\Phi^{-1}(t; t_0) = \Phi(t_0; t).$$

Running the system backward in time is the same as inverting the forward map.

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The Non-Homogeneous Case

Now we add a forcing term and solve

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t).$$

Let $\Phi(t; t_0, X_0)$ be any fundamental matrix of the homogeneous problem. Then $\mathbf{x} = \Phi(t; t_0, X_0)\mathbf{c}$ solves the homogeneous problem for every constant vector $\mathbf{c}$.

Idea. Use the *variation of parameters* technique: replace the constant vector $\mathbf{c}$ by an unknown function $\mathbf{v}(t)$, so we look for a solution of the form

$$\mathbf{x} = \Phi(t; t_0, X_0)\mathbf{v}(t).$$

Substituting this guess into the non-homogeneous equation and using the product rule,

$$\begin{aligned}\mathbf{x}' &= \Phi'(t; t_0, X_0)\mathbf{v}(t) + \Phi(t; t_0, X_0)\mathbf{v}'(t) \\ &= A(t) \underbrace{\Phi(t; t_0, X_0)\mathbf{v}(t)}_{\mathbf{x}} + \Phi(t; t_0, X_0)\mathbf{v}'(t)\end{aligned}$$

$$= A(t)\mathbf{x} + \Phi(t; t_0, X_0)\mathbf{v}'(t).$$

Comparing with the original equation, the extra term must equal the forcing:

$$\begin{aligned} \Rightarrow \Phi(t; t_0, X_0)\mathbf{v}'(t) &= \mathbf{f}(t) \\ \Rightarrow \mathbf{v}'(t) &= \Phi^{-1}(t; t_0, X_0)\mathbf{f}(t) \\ \Rightarrow \mathbf{v}(t) &= \underbrace{\int \Phi^{-1}(t; t_0, X_0)\mathbf{f}(t) dt}_{\mathbf{V}(t)} + \mathbf{c} = \mathbf{V}(t) + \mathbf{c}. \end{aligned}$$

Hence the general solution is the sum of a particular part and a complementary part:

$$\mathbf{x} = \Phi(t; t_0, X_0) \cdot (\mathbf{V}(t) + \mathbf{c}) = \underbrace{\phi_p(t)}_{\text{particular solution}} + \underbrace{\phi_c(t)}_{\text{complementary solution}}.$$

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If an initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$ is given, we choose $\Phi(t; t_0, X_0)$ to be the state-transition matrix and integrate from t_0 :

$$\mathbf{V}(t) = \int_{t_0}^t \Phi^{-1}(\tau; t_0)\mathbf{f}(\tau) d\tau = \int_{t_0}^t \Phi(t_0; \tau)\mathbf{f}(\tau) d\tau.$$

Then $\mathbf{V}(t_0) = \mathbf{0}$, so $\mathbf{c}$ must be $\mathbf{x}_0$. Using $\Phi(t; t_0)\Phi(t_0; \tau) = \Phi(t; \tau)$ from rule (ii), the solution becomes:

Solution of the non-homogeneous problem

$$\mathbf{x}(t) = \underbrace{\Phi(t; t_0)\mathbf{x}_0}_{\substack{\phi_0(t) \\ \text{due to initial condition}}} + \underbrace{\int_{t_0}^t \Phi(t; \tau)\mathbf{f}(\tau) d\tau}_{\substack{\phi_f(t) \\ \text{due to input}}}.$$

The first term is the response of the system to its initial state alone. The second adds up the response to the input: the input applied at time τ is carried to time t by $\Phi(t; \tau)$.

ex/ Find the solution of the initial value problem

$$\mathbf{x}' = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(t), \quad \mathbf{x}(0) = \mathbf{0}, \quad u(t) = \begin{cases} 1 & t > 0 \\ 0 & t \leq 0. \end{cases}$$

First we need a fundamental matrix for the homogeneous problem. The two functions

$$\phi_1 = \begin{bmatrix} \cos t \\ \cos t - \sin t \end{bmatrix}, \quad \phi_2 = \begin{bmatrix} \sin t \\ \cos t + \sin t \end{bmatrix}$$

are linearly independent solutions, so a fundamental matrix is

$$\Phi(t) = [\phi_1(t) \ \phi_2(t)] = \begin{bmatrix} \cos t & \sin t \\ \cos t - \sin t & \cos t + \sin t \end{bmatrix}.$$

The state-transition matrix is then $\Phi(t; t_0) = \Phi(t)\Phi^{-1}(t_0)$, which evaluates to

$$\Phi(t; t_0) = \begin{bmatrix} \cos t & \sin t \\ \cos t - \sin t & \cos t + \sin t \end{bmatrix} \begin{bmatrix} \cos t_0 + \sin t_0 & -\sin t_0 \\ -\cos t_0 + \sin t_0 & \cos t_0 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(t - t_0) - \sin(t - t_0) & \sin(t - t_0) \\ -2\sin(t - t_0) & \cos(t - t_0) + \sin(t - t_0) \end{bmatrix}.$$

As expected for a time-invariant system, it depends only on $t - t_0$.

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For $t > 0$ we have $u(t) = 1$, so $\mathbf{f}(t) = (1, 2)^T$. We solve by variation of parameters, with the fundamental matrix $\Phi(t)$ found above (any fundamental matrix works):

$$\mathbf{x} = \Phi(t)\mathbf{v}(t).$$

$$\begin{aligned} \mathbf{v}'(t) &= \Phi^{-1}(t)\mathbf{f}(t) = \begin{bmatrix} \cos t + \sin t & -\sin t \\ -\cos t + \sin t & \cos t \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad t > 0 \\ &= \begin{bmatrix} \cos t - \sin t \\ \cos t + \sin t \end{bmatrix}. \end{aligned}$$

$$\Rightarrow \mathbf{v}(t) = \begin{bmatrix} \sin t + \cos t + c_1 \\ \sin t - \cos t + c_2 \end{bmatrix}.$$

Multiplying by $\Phi(t)$ gives the general solution:

$$\mathbf{x} = \Phi(t)\mathbf{v}(t) = \begin{bmatrix} 1 + c_1 \cos t + c_2 \sin t \\ (c_1 + c_2) \cos t + (c_2 - c_1) \sin t \end{bmatrix}.$$

We impose the initial condition to find c_1 and c_2 :

$$\mathbf{x}(0) = \mathbf{0} \Rightarrow \begin{bmatrix} 1 + c_1 \\ c_1 + c_2 \end{bmatrix} = \mathbf{0} \Rightarrow c_1 = -1, \quad c_2 = 1.$$

Hence the solution is

Solution of the example

$$\mathbf{x}(t) = \begin{bmatrix} 1 - \cos t + \sin t \\ 2 \sin t \end{bmatrix}, \quad t \geq 0.$$

A Second Way: the Formula Directly

Since $\mathbf{x}(0) = \mathbf{0}$, the initial-condition term vanishes and the input term gives the answer in one step, with no unknown constants to fix:

$$\begin{aligned} \mathbf{x}(t) &= \int_0^t \Phi(t; \tau) \mathbf{f}(\tau) d\tau \\ &= \int_0^t \begin{bmatrix} \cos(t - \tau) - \sin(t - \tau) & \sin(t - \tau) \\ -2\sin(t - \tau) & \cos(t - \tau) + \sin(t - \tau) \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} d\tau \\ &= \int_0^t \begin{bmatrix} \cos(t - \tau) + \sin(t - \tau) \\ 2 \cos(t - \tau) \end{bmatrix} d\tau = \begin{bmatrix} 1 - \cos t + \sin t \\ 2 \sin t \end{bmatrix}, \end{aligned}$$

which agrees with the result above.

— extra note: Linear Dynamical Systems, page 1 (printed 97) —

Supplement supplied separately; wording and calculations retained as written.

ex/ A time-varying system with fixed eigenvectors. Consider the homogeneous, time-varying linear system of differential equations

$$\mathbf{x}' = A(t)\mathbf{x} \quad (\text{Eq.1})$$

where $A(t)$ is $n \times n$ and has the form $A(t) = V \underbrace{\Lambda(t)}_{\text{diagonal}} V^{-1}$. That is, for every t the matrix $A(t)$

has the same eigenvectors, but the corresponding eigenvalues are functions of time.

In this special case we can decouple the system. Define the new unknown $\mathbf{z} = V^{-1}\mathbf{x}$, and substitute $\mathbf{x} = V\mathbf{z}$ into (Eq.1). Because V is constant, $\mathbf{x}' = V\mathbf{z}'$, so

$$\begin{aligned} V\mathbf{z}' &= V\Lambda(t)V^{-1}V\mathbf{z} \\ \Rightarrow \mathbf{z}' &= \Lambda(t)\mathbf{z} \quad (\text{Eqn.2}). \end{aligned}$$

Since $\Lambda(t)$ is diagonal, (Eqn.2) is a set of n separate first-order equations:

$$z'_k = \lambda_k(t)z_k, \quad 1 \leq k \leq n.$$

Each one is solved by separating variables:

$$\begin{aligned} \frac{dz_k}{dt} &= \lambda_k(t)z_k \Rightarrow \int_{z_k(0)}^{z_k(t)} \frac{dz_k}{z_k} = \int_0^t \lambda_k(\tau) d\tau \\ \Rightarrow \ln |z_k| &= \int_0^t \lambda_k(\tau) d\tau + d_k. \end{aligned}$$

Exponentiating gives

Decoupled solution

$$z_k(t) = C_k \cdot e^{\int_0^t \lambda_k(\tau) d\tau}.$$

Returning to the original unknown, with $\mathbf{v}_k$ the columns of V ,

$$\mathbf{x}(t) = V\mathbf{z}(t) = \sum_{k=1}^n \mathbf{v}_k z_k(t) = \sum_{k=1}^n \mathbf{v}_k C_k e^{\int_0^t \lambda_k(\tau) d\tau}.$$

Note that if the system starts at an eigenvector, $\mathbf{x}(0) = \mathbf{v}_m$, then $C_k = 0$ for $k \neq m$ and $C_m = 1$, so the solution stays on the line through $\mathbf{v}_m$. Collecting these n solutions as columns gives a fundamental matrix:

$$\underbrace{\Phi(t; 0, V)}_{\text{fundamental matrix}} = [\mathbf{v}_1 e^{\int_0^t \lambda_1(\tau) d\tau} \quad \dots \quad \mathbf{v}_n e^{\int_0^t \lambda_n(\tau) d\tau}].$$

— extra note: Linear Dynamical Systems, page 2 (printed 98) —

Supplement supplied separately; wording and calculations retained as written.

The same fundamental matrix can be written as a product with a diagonal matrix:

$$\Phi(t; 0, V) = V \begin{bmatrix} e^{\int_0^t \lambda_1(\tau) d\tau} & & \\ & \ddots & \\ & & e^{\int_0^t \lambda_n(\tau) d\tau} \end{bmatrix}.$$

The state-transition matrix is obtained by normalizing with V^{-1} :

$$\Phi(t; 0) = \Phi(t; 0, V)V^{-1} = V \begin{bmatrix} e^{\int_0^t \lambda_1(\tau) d\tau} & & \\ & \ddots & \\ & & e^{\int_0^t \lambda_n(\tau) d\tau} \end{bmatrix} V^{-1}.$$

To start at a general time t_0 we use the composition and inverse rules of the previous section, $\Phi(t; t_0) = \Phi(t; 0)\Phi^{-1}(t_0; 0)$. The middle factors $V^{-1}V$ cancel:

$$\begin{aligned}\Phi(t; t_0) &= \Phi(t; 0)\Phi^{-1}(t_0; 0) \\ &= V \begin{bmatrix} e^{\int_0^t \lambda_1(\tau) d\tau} & & \\ & \ddots & \\ & & e^{\int_0^t \lambda_n(\tau) d\tau} \end{bmatrix} V^{-1} V \begin{bmatrix} e^{-\int_0^{t_0} \lambda_1(\tau) d\tau} & & \\ & \ddots & \\ & & e^{-\int_0^{t_0} \lambda_n(\tau) d\tau} \end{bmatrix} V^{-1}.\end{aligned}$$

Combining the exponentials, the two integrals merge into one from t_0 to t :

STM for constant eigenvectors

$$\Phi(t; t_0) = V \begin{bmatrix} e^{\int_{t_0}^t \lambda_1(\tau) d\tau} & & \\ & \ddots & \\ & & e^{\int_{t_0}^t \lambda_n(\tau) d\tau} \end{bmatrix} V^{-1} \quad (\text{Eqn.3}).$$

ex/ Find the state-transition matrix of

$$\mathbf{x}' = \underbrace{\begin{bmatrix} 1-t & t/2-1 \\ 0 & -t/2 \end{bmatrix}}_{A(t)} \mathbf{x}.$$

Since $A(t)$ is upper triangular, its eigenvalues are the diagonal entries, $\lambda_1 = 1-t$ and $\lambda_2 = -\frac{t}{2}$. We find the eigenvectors from $(A - \lambda I)\mathbf{v} = \mathbf{0}$:

$$(A - \lambda_1 I)\mathbf{v}_1 = \mathbf{0} \Rightarrow \begin{bmatrix} 0 & t/2-1 \\ 0 & t/2-1 \end{bmatrix} \underbrace{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}}_{\mathbf{v}_1} = \mathbf{0} \Rightarrow \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$(A - \lambda_2 I)\mathbf{v}_2 = \mathbf{0} \Rightarrow \begin{bmatrix} 1-t/2 & t/2-1 \\ 0 & 0 \end{bmatrix} \mathbf{v}_2 = \mathbf{0} \Rightarrow \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Both eigenvectors are constants, which is what (Eqn.3) requires.

— extra note: Linear Dynamical Systems, page 3 (printed 99) —

Supplement supplied separately; wording and calculations retained as written.

Therefore $A(t)$ factors as

$$\begin{aligned}A(t) &= [\mathbf{v}_1 \ \mathbf{v}_2] \begin{bmatrix} \lambda_1(t) & \\ & \lambda_2(t) \end{bmatrix} [\mathbf{v}_1 \ \mathbf{v}_2]^{-1} \\ &= \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_V \underbrace{\begin{bmatrix} 1-t & 0 \\ 0 & -t/2 \end{bmatrix}}_{\Lambda(t)} \underbrace{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}}_{V^{-1}}.\end{aligned}$$

By (Eqn.3) of the previous page, the state-transition matrix is

$$\Phi(t; t_0) = V \begin{bmatrix} e^{\int_{t_0}^t \lambda_1(\tau) d\tau} & 0 \\ 0 & e^{\int_{t_0}^t \lambda_2(\tau) d\tau} \end{bmatrix} V^{-1}.$$

The two integrals are

$$\int_{t_0}^t \lambda_1(\tau) d\tau = \int_{t_0}^t (1-\tau) d\tau = \tau - \frac{1}{2}\tau^2 \Big|_{t_0}^t = (t-t_0) - \frac{1}{2}(t^2 - t_0^2),$$

$$\int_{t_0}^t \lambda_2(\tau) d\tau = \int_{t_0}^t -\frac{\tau}{2} d\tau = -\frac{\tau^2}{4} \Big|_{t_0}^t = -\frac{1}{4}(t^2 - t_0^2).$$

Thus

$$\begin{aligned} \Phi(t; t_0) &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{(t-t_0)-\frac{1}{2}(t^2-t_0^2)} & 0 \\ 0 & e^{-\frac{1}{4}(t^2-t_0^2)} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{(t-t_0)-\frac{1}{2}(t^2-t_0^2)} & -e^{(t-t_0)-\frac{1}{2}(t^2-t_0^2)} \\ 0 & e^{-\frac{1}{4}(t^2-t_0^2)} \end{bmatrix}. \end{aligned}$$

Carrying out the last product gives the final answer:

State-transition matrix of the example

$$\Phi(t; t_0) = \begin{bmatrix} e^{(t-t_0)-\frac{1}{2}(t^2-t_0^2)} & -e^{(t-t_0)-\frac{1}{2}(t^2-t_0^2)} + e^{-\frac{1}{4}(t^2-t_0^2)} \\ 0 & e^{-\frac{1}{4}(t^2-t_0^2)} \end{bmatrix}.$$

— extra note: STATE TRANS MATRIX, page 1 —

Supplement supplied separately; wording and calculations retained as written.

How to Find the State-Transition Matrix

This page collects the standard ways to find the state-transition matrix (STM) $\Phi(t; t_0)$, first for time-invariant systems $\mathbf{x}' = A\mathbf{x}$, then for a time-varying case.

Time-Invariant Homogeneous Equations

Method 1: diagonalization (or the Cayley–Hamilton method). If $A = Q\Lambda Q^{-1}$, the STM is the matrix exponential, which is computed on the eigenvalues:

STM by diagonalization

$$\Phi(t; 0) = e^{At} = Q \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix} Q^{-1}.$$

Remark: For a time-invariant system, $\Phi(t; t_0) = \Phi(t - t_0; 0) = e^{A(t-t_0)}$.

Method 2: inverse Laplace transform. Transforming $\mathbf{x}' = A\mathbf{x}$ gives $\bar{\mathbf{x}} = (sI - A)^{-1}\mathbf{x}_0$, so

STM by the inverse Laplace transform

$$\Phi(t; 0) = \mathcal{L}^{-1} \{ (sI - A)^{-1} \}.$$

Method 3: from a fundamental matrix. Find

$$\Phi(t; t_0, X_0) = [\phi(t; t_0, \mathbf{x}_1) \cdots \phi(t; t_0, \mathbf{x}_n)]$$

by building the solutions from the eigenpairs $(\lambda_k, \mathbf{u}_k)$; for $n = 2$ these are combinations $C_1 \mathbf{u}_1 e^{\lambda_1 t} + C_2 \mathbf{u}_2 e^{\lambda_2 t}$. Then normalize with the inverse of the starting matrix:

$$\Phi(t; t_0) = \Phi(t; t_0, X_0) \cdot X_0^{-1}.$$

Time-Varying Homogeneous Equations

Diagonalization with constant eigenvectors. If $A(t) = Q\Lambda(t)Q^{-1}$ with a constant eigenvector matrix Q , then each eigenvalue is integrated separately:

$$\Phi(t; t_0) = Q \begin{bmatrix} e^{\int_{t_0}^t \lambda_1(\tau) d\tau} & & \\ & \ddots & \\ & & e^{\int_{t_0}^t \lambda_n(\tau) d\tau} \end{bmatrix} Q^{-1}.$$

Remark: This is valid only when the eigenvector matrix Q does not change with time; see the Linear Dynamical Systems notes for the derivation.

Time-Invariant Non-Homogeneous Equations

For $\mathbf{x}' = A\mathbf{x} + B\mathbf{u}(t)$ with $\mathbf{x}(t_0) = \mathbf{x}_0$, the solution $\phi(t; t_0, \mathbf{x}_0)$ combines the response to the initial state with the response to the input:

Time-invariant forced response

$$\phi(t; t_0, \mathbf{x}_0) = e^{A(t-t_0)}\mathbf{x}_0 + e^{At} \int_{t_0}^t e^{-A\tau} B\mathbf{u}(\tau) d\tau.$$