

Section 7: Quadratic Forms

MATH241 Lecture Notes, notes pages 65–68

Notation. Scalars are plain italic (a, α, z), vectors are bold ($\mathbf{u}, \mathbf{x}$), and matrices are italic capitals (A, B ; zero matrix O , identity I). The handwritten notes underline vectors once and matrices twice.

These notes are built on top of Orhan Arıkan's linear algebra course notes.

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Quadratic Forms

A *quadratic form* is a polynomial in which every term has degree two. In two variables it looks like

$$f(x_1, x_2) = a_{11}x_1^2 + a_{22}x_2^2 + 2a_{12}x_1x_2.$$

With n variables, a quadratic form can always be written in matrix form, and the matrix A can be chosen to be symmetric:

Quadratic form

$$f(x_1, \dots, x_n) = \mathbf{x}^T A \mathbf{x}.$$

Def'n A quadratic form is called a *canonical* quadratic form if it has no cross terms, that is, if its matrix is diagonal:

$$f(x_1, \dots, x_n) = \mathbf{x}^T D \mathbf{x}, \quad D \text{ is diagonal.}$$

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Theorem 11.6.1

Every quadratic form can be brought to canonical form by a change of variables.

Theorem 11.6.1

Every quadratic form can be transformed into a canonical form.

proof Let $f(x_1, \dots, x_n) = \mathbf{x}^T A \mathbf{x}$ with A symmetric. A symmetric matrix can be diagonalized by an orthogonal matrix Q whose columns are unit-norm eigenvectors:

$$Q^T A Q = \Lambda, \quad \text{so} \quad A = Q \Lambda Q^T.$$

Then, introducing the new variables $\tilde{\mathbf{x}} = Q^T \mathbf{x}$ (equivalently $\mathbf{x} = Q \tilde{\mathbf{x}}$),

$$\begin{aligned} \underbrace{f(x_1, \dots, x_n)}_{f(\mathbf{x})} &= \mathbf{x}^T Q \Lambda Q^T \mathbf{x} \\ &= (Q^T \mathbf{x})^T \Lambda (Q^T \mathbf{x}) \\ &= \tilde{\mathbf{x}}^T \Lambda \tilde{\mathbf{x}}. \end{aligned}$$

Thus $f(Q\tilde{\mathbf{x}}) = \tilde{\mathbf{x}}^T \Lambda \tilde{\mathbf{x}}$, which has no cross terms, so the form is in canonical form in the new variables. ■

Def'n A quadratic form $\mathbf{x}^T A \mathbf{x}$ is *positive definite* if $\mathbf{x}^T A \mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$, and *negative definite* if $\mathbf{x}^T A \mathbf{x} < 0$ for all $\mathbf{x} \neq \mathbf{0}$. A symmetric matrix A is called positive (negative) definite if its quadratic form is.

Theorem 11.6.2

The eigenvalues tell us the sign of the form.

Theorem 11.6.2

For a real symmetric A , A is positive (negative) definite if and only if every eigenvalue of A is positive (negative).

proof Let Q have unit-norm eigenvectors as its columns, so that $A = Q\Lambda Q^T$. Then, with $\tilde{x} = Q^T x$,

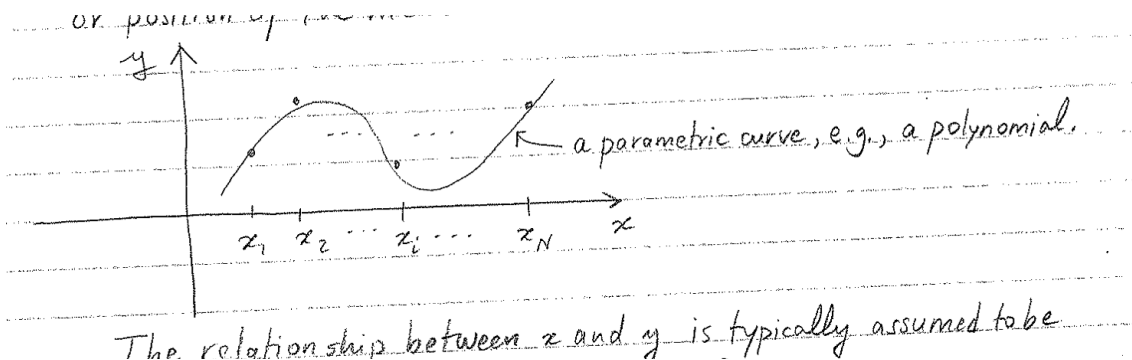
$$\begin{aligned} x^T A x &= x^T (Q \Lambda Q^T) x \\ &= (Q^T x)^T \Lambda (Q^T x) \\ &= \tilde{x}^T \Lambda \tilde{x} \\ &= \sum_{i=1}^n \lambda_i \tilde{x}_i^2 \quad \begin{cases} > 0 & \text{if all } \lambda_i > 0 \\ < 0 & \text{if all } \lambda_i < 0. \end{cases} \end{aligned}$$

Here $\tilde{x} \neq 0$ whenever $x \neq 0$, so at least one term is non-zero and the sign of the sum is the common sign of the eigenvalues. Conversely, if A is positive definite and (λ, x) is an eigenpair with $x^T x = 1$, then $\lambda = \lambda x^T x = x^T A x > 0$; the negative definite case is the same. ■

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Curve Fitting

In many applications we are given a set of measurements (x_i, y_i) , $1 \leq i \leq N$, where x_i may be the time or the position of the i^{th} measurement.



The relationship between x and y is usually assumed to be a parametric function, with a vector θ of parameters:

$$y = f(x; \underbrace{\theta}_{\text{the parameters, in vector form}}).$$

We then choose the parameters that fit the measurements best. A common choice is the parameters that minimize the sum of the squared errors:

$$\sum_{i=1}^N (y_i - f(x_i; \theta))^2, \quad \text{or} \quad \theta^* = \arg \min_{\theta} \sum_{i=1}^N (y_i - f(x_i; \theta))^2.$$

In general this is solved with numerical optimization tools, such as those available in MATLAB. But in some cases it reduces to linear algebra. This happens when f is a polynomial in x :

$$f(x; \theta) = \theta_n x^n + \theta_{n-1} x^{n-1} + \cdots + \theta_1 x + \theta_0.$$

Let e_i be the error at the point x_i :

$$e_i = y_i - (\theta_n x_i^n + \theta_{n-1} x_i^{n-1} + \cdots + \theta_1 x_i + \theta_0).$$

Collecting all the errors in a vector gives a matrix equation, where X has one row $[1, x_i, x_i^2, \dots, x_i^n]$ per measurement:

$$\underbrace{\begin{bmatrix} e_1 \\ \vdots \\ e_N \end{bmatrix}}_{\mathbf{e}} = \underbrace{\begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}}_{\mathbf{y}} - \underbrace{\begin{bmatrix} 1 & x_1 & \cdots & x_1^n \\ \vdots & & & \vdots \\ 1 & x_N & \cdots & x_N^n \end{bmatrix}}_X \underbrace{\begin{bmatrix} \theta_0 \\ \vdots \\ \theta_n \end{bmatrix}}_{\boldsymbol{\theta}}.$$

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So the optimization problem is to choose $\boldsymbol{\theta}$ to make the error vector as short as possible:

$$\min_{\boldsymbol{\theta}} \|\mathbf{e}\|^2 = \min_{\boldsymbol{\theta}} \|\mathbf{y} - X\boldsymbol{\theta}\|^2.$$

This is a best-approximation problem like the one we solved earlier: we look for the vector $X\boldsymbol{\theta}$ in the column space of X that is closest to $\mathbf{y}$. The answer is characterized by orthogonality: at the optimal $\boldsymbol{\theta}$ the error $\mathbf{e}$ must be orthogonal to the column space of X , that is, $X^T \mathbf{e} = \mathbf{0}$. Substituting $\mathbf{e} = \mathbf{y} - X\boldsymbol{\theta}$:

The normal equations

$$X^T \mathbf{e} = \mathbf{0} \Rightarrow X^T (\mathbf{y} - X\boldsymbol{\theta}) = \mathbf{0} \Rightarrow X^T X \boldsymbol{\theta} = X^T \mathbf{y}.$$

The normal equations are always consistent, because the closest point to $\mathbf{y}$ in the column space of X always exists (it is the orthogonal projection of $\mathbf{y}$), and it has the form $X\boldsymbol{\theta}$ for some $\boldsymbol{\theta}$. So they can be solved for the optimal parameters.

ex/ Fit a second-order polynomial to the data $\mathbf{y} = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 2 \end{bmatrix}$ measured at $x_1 = 0, x_2 = 1, x_3 = 2, x_4 = 3$. There are three parameters $\theta_0, \theta_1, \theta_2$, and the normal equations are

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}}_{X^T} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}}_X \underbrace{\begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \end{bmatrix}}_{\boldsymbol{\theta}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}}_{X^T} \underbrace{\begin{bmatrix} 1 \\ -1 \\ 3 \\ 2 \end{bmatrix}}_{\mathbf{y}}$$

Multiplying out,

$$\begin{bmatrix} 4 & 6 & 14 \\ 6 & 14 & 36 \\ 14 & 36 & 98 \end{bmatrix} \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ 29 \end{bmatrix}.$$

Solving this system gives the best-fit parameters

$$\boldsymbol{\theta} = \begin{bmatrix} 9/20 \\ -1/20 \\ 1/4 \end{bmatrix}.$$