

Section 10: Laplace Transform

MATH241 Lecture Notes, notes pages 84–89

Notation. Scalars are plain italic (a, α, z), vectors are bold ($\mathbf{u}, \mathbf{x}$), and matrices are italic capitals (A, B ; zero matrix O , identity I). The handwritten notes underline vectors once and matrices twice.

These notes are built on top of Orhan Arikan's linear algebra course notes.

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Laplace Transform*

(* Chapter 5 in the textbook.)

A general linear integral transform has the form

$$F(s) = \int_a^b K(t, s) f(t) dt,$$

where $K(t, s)$ is called the *transformation kernel*. The Laplace transform is the case with kernel e^{-st} on the interval $[0, \infty)$, where s is a complex-valued variable of the transform domain:

The Laplace transform

$$F(s) = \int_0^\infty f(t) e^{-st} dt.$$

Just as the z-transform reduces difference equations to linear algebraic equations, the Laplace transform reduces linear constant-coefficient differential equations to linear algebraic equations.

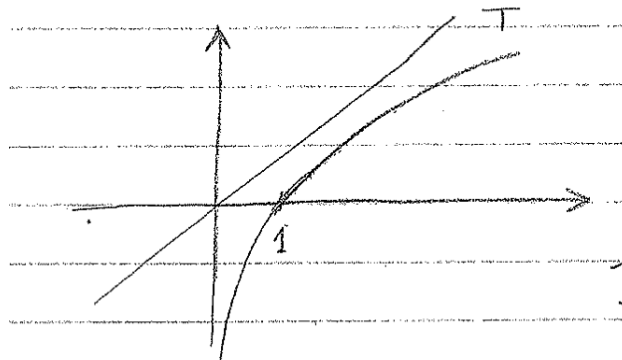
Calculation of the transform

The integral $F(s) = \int_0^\infty f(t) e^{-st} dt$ does not converge for every function. Convergence is assured if f is of *exponential order*, that is, if there are constants K and c with

$$|f(t)| \leq K e^{ct} \quad \text{for } t > T.$$

ex/ $f(t) = \sin(\omega t)$ is of exponential order, because $|f(t)| \leq e^{0t} = 1$ for $t > 0$.

ex/ $f(t) = t^2$ is of exponential order, because $t^2 \leq e^t$ for $t > T$. The condition $T^2 \leq e^T$ is the same as $2 \ln T \leq T$, which holds for every $T > 0$, as the picture shows: the line $y = T$ stays above the curve $y = 2 \ln T$.



We can also check this with l'Hôpital's rule:

$$\lim_{t \rightarrow \infty} \frac{t^2}{e^t} = \lim_{t \rightarrow \infty} \frac{2t}{e^t} = \lim_{t \rightarrow \infty} \frac{2}{e^t} = 0.$$

l'Hôpital's rule

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ex/ A function that is not of exponential order: $f(t) = e^{t^2}$. It grows faster than any exponential e^{ct} , whatever c we choose:

$$\lim_{t \rightarrow \infty} \frac{e^{t^2}}{e^{ct}} = \lim_{t \rightarrow \infty} e^{(t^2 - ct)} = \infty, \quad \text{for any } c.$$

Theorem 5.2.1

The next theorem says when the Laplace transform exists.

Theorem 5.2.1: existence of the Laplace transform

Let $f(t)$ satisfy

- (i) $f(t)$ is piecewise continuous on every interval $0 \leq t \leq A$, with a finite number of discontinuities, and
- (ii) $f(t)$ is of exponential order: $|f(t)| \leq Ke^{ct}$ for $t \geq T$.

Then the Laplace transform $F(s)$ exists for $\Re\{s\} > c$.

proof Split the integral at T . The first part is over a finite interval, and it exists for every s because of condition (i):

$$F(s) = \int_0^\infty f(t)e^{-st} dt = \underbrace{\int_0^T f(t)e^{-st} dt}_{\text{exists for all } s \text{ by (i)}} + \int_T^\infty f(t)e^{-st} dt.$$

For the second part, we bound the integrand using condition (ii) and $|e^{-st}| = e^{-\Re\{s\}t}$:

$$\begin{aligned} \left| \int_T^\infty f(t)e^{-st} dt \right| &\leq \int_T^\infty |f(t)e^{-st}| dt \leq \int_T^\infty |f(t)| |e^{-st}| dt \\ &\leq \int_T^\infty Ke^{ct} e^{-\Re\{s\}t} dt = K \int_T^\infty e^{-(\Re\{s\} - c)t} dt \\ &= K \frac{e^{-(\Re\{s\} - c)T}}{\Re\{s\} - c} < \infty \end{aligned}$$

for $\Re\{s\} > c$, so the second part converges absolutely. ■

ex/ The exponential $f(t) = e^{at}$:

$$F(s) = \int_0^\infty e^{at} e^{-st} dt = \int_0^\infty e^{-(s-a)t} dt = \frac{1}{s-a}, \quad \Re\{s\} > a.$$

ex/ The sine $f(t) = \sin at$. Write the sine with complex exponentials and use the previous example twice:

$$F(s) = \int_0^\infty \sin at e^{-st} dt = \int_0^\infty \left(\frac{e^{jat} - e^{-jat}}{2j} \right) e^{-st} dt$$

$$\begin{aligned}
 &= \frac{1}{2j} \int_0^\infty e^{-(s-ja)t} dt - \frac{1}{2j} \int_0^\infty e^{-(s+ja)t} dt \\
 &= \frac{1}{2j} \frac{1}{s-ja} - \frac{1}{2j} \frac{1}{s+ja} = \frac{1}{2j} \frac{(s+ja) - (s-ja)}{s^2 + a^2} = \frac{a}{s^2 + a^2}, \quad \Re\{s\} > 0.
 \end{aligned}$$

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Notes

(i) The inverse Laplace transform is also an integral transform, but computing it directly requires techniques from complex analysis. In these notes we therefore invert by inspection, using known transform pairs, as we did for the z-transform.

(ii) The inverse is unique: if

$$f_1(t) \xleftrightarrow{\mathcal{L}} F_1(s), \quad f_2(t) \xleftrightarrow{\mathcal{L}} F_2(s)$$

and $\int_0^\infty |f_1(t) - f_2(t)| dt > 0$, then $F_1(s) \neq F_2(s)$. (Functions that differ only at isolated points have the same transform, which is why the condition is stated with an integral.)

Properties of the Laplace Transform

These properties are what make the Laplace transform useful for differential equations.

(i) *Linearity*. For all constants α, β and all functions $u(t)$ and $v(t)$ of exponential order,

$$\mathcal{L}\{\alpha u(t) + \beta v(t)\} = \alpha \mathcal{L}\{u(t)\} + \beta \mathcal{L}\{v(t)\}.$$

(ii) *Linearity of the inverse*. For all constants α, β and transforms $U(s)$ and $V(s)$,

$$\mathcal{L}^{-1}\{\alpha U(s) + \beta V(s)\} = \alpha \mathcal{L}^{-1}\{U(s)\} + \beta \mathcal{L}^{-1}\{V(s)\}.$$

(iii) *Derivative*. Differentiation in time becomes multiplication by s , with the initial value subtracted:

Derivative property

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0).$$

proof Integrate by parts:

$$\begin{aligned}
 \mathcal{L}\{f'(t)\} &= \int_0^\infty f'(t)e^{-st} dt = f(t)e^{-st} \Big|_0^\infty + s \int_0^\infty f(t)e^{-st} dt \\
 &= -f(0) + s\mathcal{L}\{f(t)\}.
 \end{aligned}$$

(The property remains valid even if $f'(t)$ is only piecewise continuous.)

(iv) *Higher derivatives*. Applying the rule twice gives the second derivative:

$$\begin{aligned}
 \mathcal{L}\{f''(t)\} &= s\mathcal{L}\{f'(t)\} - f'(0) \\
 &= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0).
 \end{aligned}$$

In general,

$$\Rightarrow \mathcal{L}\{f^{(k)}(t)\} = s^k \mathcal{L}\{f(t)\} - \sum_{i=1}^k s^{k-i} f^{(i-1)}(0).$$

This is useful for solving differential equations with given initial conditions, since the initial conditions appear automatically.

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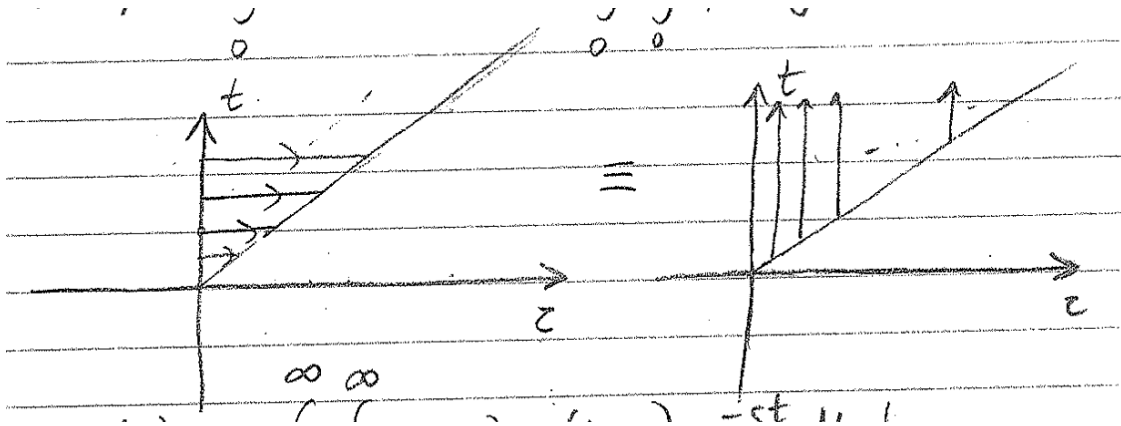
(v) *Convolution*. For functions $f(t)$ and $g(t)$ that are zero for $t < 0$, the *convolution* is

$$h(t) = f(t) * g(t) = (f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau.$$

Its Laplace transform is the double integral

$$H(s) = \int_0^\infty h(t)e^{-st} dt = \int_0^\infty \int_0^t f(\tau)g(t - \tau) d\tau e^{-st} dt.$$

The key step is to swap the order of integration. The region of integration, $0 \leq \tau \leq t < \infty$, is the same whether we sum along horizontal slices (τ from 0 to t) or along vertical slices (t from τ to ∞):



After the swap, we substitute $\mu = t - \tau$ in the inner integral, and the double integral splits into a product of two single integrals:

$$\begin{aligned} \Rightarrow H(s) &= \int_0^\infty \int_\tau^\infty f(\tau)g(t - \tau)e^{-st} dt d\tau \\ &= \int_0^\infty f(\tau) \left[\int_\tau^\infty \underbrace{g(t - \tau)}_\mu e^{-st} dt \right] d\tau \\ &= \int_0^\infty f(\tau) \left[\int_0^\infty g(\mu)e^{-s(\tau + \mu)} d\mu \right] d\tau \\ &= \int_0^\infty f(\tau) \left[\int_0^\infty g(\mu)e^{-s\mu} d\mu \right] e^{-s\tau} d\tau \\ &= \left[\int_0^\infty f(\tau)e^{-s\tau} d\tau \right] \left[\int_0^\infty g(\mu)e^{-s\mu} d\mu \right]. \end{aligned}$$

Convolution theorem

$$H(s) = F(s) \cdot G(s).$$

So convolution in time becomes ordinary multiplication in the s -domain.

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ex/ Find the inverse transform of $F(s) = \frac{3}{s^2 + 3s - 10} = \frac{3}{(s+5)(s-2)}$. There are two ways.

(i) *Partial fractions.*

$$\frac{3}{(s+5)(s-2)} = \frac{3}{7} \left[\frac{1}{s-2} - \frac{1}{s+5} \right] \Rightarrow f(t) = \frac{3}{7}(e^{2t} - e^{-5t}), \quad t \geq 0.$$

(ii) *Convolution.* Write $F(s)$ as a product of two simple transforms, and use the convolution property:

$$\begin{aligned} \frac{3}{(s+5)(s-2)} &= 3 \cdot \frac{1}{s+5} \cdot \frac{1}{s-2} \\ \Rightarrow f(t) &= 3 \cdot (e^{-5t} * e^{2t}) \\ &= 3 \cdot \int_0^t e^{2\tau} e^{-5(t-\tau)} d\tau \\ &= 3e^{-5t} \int_0^t e^{7\tau} d\tau = 3e^{-5t} \frac{1}{7} (e^{7\tau} \Big|_0^t) \\ &= \frac{3}{7} e^{-5t} (e^{7t} - 1) \\ &= \frac{3}{7} (e^{2t} - e^{-5t}), \quad t \geq 0. \end{aligned}$$

Both methods give the same answer.

Solution to Linear Constant Coefficient Differential Equations by Laplace Transform

Transform the differential equation, solve the resulting algebraic equation, then invert. As an example take $x'' + ax' + bx = f(t)$ and transform both sides with the derivative property, writing $\bar{X}(s)$ for the transform of x :

$$\begin{aligned} (s^2 \bar{X}(s) - sx(0) - x'(0)) + a(s\bar{X}(s) - x(0)) + b\bar{X}(s) &= F(s). \\ \Rightarrow (s^2 + as + b)\bar{X}(s) &= sx(0) + ax(0) + x'(0) + F(s). \end{aligned}$$

Dividing by $s^2 + as + b$ shows that the solution has two parts: one due to the initial conditions, and one due to the input $f(t)$:

Solution in the s -domain

$$\bar{X}(s) = \underbrace{\frac{(s+a)x(0) + x'(0)}{s^2 + as + b}}_{\text{invert by partial fraction expansion}} + \underbrace{\frac{F(s)}{s^2 + as + b}}_{\substack{\text{invert by partial fractions} \\ \text{or by the convolution property}}}.$$

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ex/ A first-order equation with an input. Solve $x' + px = q(t)$ with $x(0) = x_0$. In the Laplace domain the derivative property turns it into an algebraic equation, with $Q(s)$ the transform of $q(t)$:

$$\begin{aligned} s\bar{X}(s) - x(0) + p\bar{X}(s) &= Q(s) \\ \Rightarrow \bar{X}(s) &= \frac{x_0}{s+p} + \frac{Q(s)}{s+p}. \end{aligned}$$

Inverting, the first term gives $x_0 e^{-pt}$ and the second term is a convolution of $q(t)$ with e^{-pt} :

$$\Rightarrow x(t) = x_0 e^{-pt} + \int_0^t q(\tau) e^{-p(t-\tau)} d\tau$$

Pulling the factor e^{-pt} out of the integral gives the form that is often called the integrating-factor solution:

Solution of $x' + px = q(t)$

$$x(t) = x_0 e^{-pt} + e^{-pt} \int_0^t q(\tau) e^{p\tau} d\tau = \left[x_0 + \int_0^t q(\tau) e^{p\tau} d\tau \right] e^{-pt}.$$