

Section 8: Linear Constant Coefficient Difference Equations

MATH241 Lecture Notes, notes pages 69–72

Notation. Scalars are plain italic (a, α, z), vectors are bold ($\mathbf{u}, \mathbf{x}$), and matrices are italic capitals (A, B ; zero matrix O , identity I). The handwritten notes underline vectors once and matrices twice.

These notes are built on top of Orhan Arikan's linear algebra course notes.

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Linear Constant Coefficient Difference Equations

A difference equation defines a sequence by a rule that gives each term from the previous ones. A linear constant-coefficient difference equation of order k has the form

$$x[n] = \underbrace{\underbrace{a_1}_{\text{coefficient}} x[n-1] + \cdots + \underbrace{a_k}_{\text{coefficient}} x[n-k]}_{k^{\text{th}}\text{-order difference equation}}$$

Such equations appear very often in discrete-time systems and in discretized models of continuous-time systems. A good way to study them is to turn the single order- k equation into a first-order equation for a vector. Stack the k most recent values into a vector $\mathbf{x}_n$. Then the next vector is obtained from the current one by a fixed matrix A , the *companion matrix*, whose first row holds the coefficients and whose other rows just shift the older values down:

$$\mathbf{x}_n = \begin{bmatrix} x[n] \\ \vdots \\ x[n-k+1] \end{bmatrix}, \quad A = \begin{bmatrix} a_1 & \cdots & a_k \\ 1 & & 0 \\ & \ddots & \vdots \\ & & 1 & 0 \end{bmatrix} = \begin{bmatrix} a_1 & \cdots & a_k \\ I_{k-1} & \mathbf{0} \end{bmatrix}.$$

Vector form

$$\mathbf{x}_n = A\mathbf{x}_{n-1}.$$

Applying this rule again and again, starting from the initial conditions $\mathbf{x}_0$, gives

$$\mathbf{x}_n = A(A(\cdots(A\mathbf{x}_0)\cdots)) = A^n\mathbf{x}_0, \quad \mathbf{x}_0 = \underbrace{\begin{bmatrix} x[0] \\ \vdots \\ x[-k+1] \end{bmatrix}}_{\text{initial conditions}}.$$

So, in principle, $\mathbf{x}_n$ for any n can be found by multiplying $\mathbf{x}_0$ by A a total of n times. This is not efficient, and it does not give a closed-form answer. Eigenvectors give more insight.

Theorem DE.1

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If $(\lambda, \mathbf{x}_0)$ is an eigenpair of A , then the solution of $\mathbf{x}_n = A\mathbf{x}_{n-1}$ that starts at $\mathbf{x}_0$ is $\mathbf{x}_n = \lambda^n \mathbf{x}_0$.

proof $\mathbf{x}_n = A^n \mathbf{x}_0 = \lambda^n \mathbf{x}_0$, since each multiplication by A just multiplies the eigenvector by λ .

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Theorem DE.2

If the starting vector is not an eigenvector, we can still handle it when A has enough eigenvectors: write $\mathbf{x}_0$ as a combination of eigenvectors and let each one evolve by itself.

Theorem DE.2

If A has a full set of eigenvectors $\mathbf{v}_i$ that span $\mathbb{R}^k$, then $\mathbf{x}_n = A^n \mathbf{x}_0$ is given by

$$\mathbf{x}_n = \sum_{i=1}^k \alpha_i \lambda_i^n \mathbf{v}_i \quad \text{where} \quad \mathbf{x}_0 = \sum_{i=1}^k \alpha_i \mathbf{v}_i.$$

proof Multiply the expansion of $\mathbf{x}_0$ by A^n , and use $A^n \mathbf{v}_i = \lambda_i^n \mathbf{v}_i$:

$$A^n \mathbf{x}_0 = A^n \left(\sum_{i=1}^k \alpha_i \mathbf{v}_i \right) = \sum_{i=1}^k \alpha_i A^n \mathbf{v}_i = \underbrace{\sum_{i=1}^k \alpha_i \lambda_i^n \mathbf{v}_i}_{\mathbf{x}_n}. \quad \blacksquare$$

Example Solve $x[n] = 2x[n-1] + x[n-2]$ with initial conditions $x[0] = 3$ and $x[-1] = 1$. Here $k = 2$ and the companion matrix has eigenvalues $1 \pm \sqrt{2}$:

$$\mathbf{x}_n = \begin{bmatrix} x[n] \\ x[n-1] \end{bmatrix}, \quad A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \underbrace{\lambda_1 = 1 + \sqrt{2}; \quad \lambda_2 = 1 - \sqrt{2}}_{\text{eigenvalues}}.$$

We find the eigenvectors by solving $(A - \lambda I)\mathbf{v} = \mathbf{0}$. For λ_1 :

$$(A - (1 + \sqrt{2})I)\mathbf{v} = \mathbf{0} \Rightarrow \begin{bmatrix} 1 - \sqrt{2} & 1 \\ 1 & -1 - \sqrt{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{v}_1 = \begin{bmatrix} 1 + \sqrt{2} \\ 1 \end{bmatrix}.$$

For λ_2 :

$$(A - (1 - \sqrt{2})I)\mathbf{v} = \mathbf{0} \Rightarrow \begin{bmatrix} 1 + \sqrt{2} & 1 \\ 1 & -1 + \sqrt{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{v}_2 = \begin{bmatrix} -1 + \sqrt{2} \\ -1 \end{bmatrix}.$$

Next we express the initial vector $\mathbf{x}_0 = \begin{bmatrix} x[0] \\ x[-1] \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ in the eigenvectors:

$$\begin{aligned} \mathbf{x}_0 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} &= \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 = \begin{bmatrix} 1 + \sqrt{2} & -1 + \sqrt{2} \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \\ \Rightarrow \alpha_2 &= \frac{2 - \sqrt{2}}{2\sqrt{2}} = \frac{\sqrt{2} - 1}{2} = -\frac{1 - \sqrt{2}}{2}, \quad \alpha_1 = 1 + \alpha_2 = \frac{\sqrt{2} + 1}{2}. \end{aligned}$$

By Theorem DE.2,

$$\begin{aligned} \mathbf{x}_n &= \alpha_1 \lambda_1^n \mathbf{v}_1 + \alpha_2 \lambda_2^n \mathbf{v}_2 \\ &= \frac{\sqrt{2} + 1}{2} (1 + \sqrt{2})^n \begin{bmatrix} 1 + \sqrt{2} \\ 1 \end{bmatrix} - \frac{1 - \sqrt{2}}{2} (1 - \sqrt{2})^n \begin{bmatrix} \sqrt{2} - 1 \\ -1 \end{bmatrix} \\ \Rightarrow \mathbf{x}_n &= \begin{bmatrix} \frac{(1 + \sqrt{2})^{n+2}}{2} + \frac{(1 - \sqrt{2})^{n+2}}{2} \\ \frac{(1 + \sqrt{2})^{n+1}}{2} + \frac{(1 - \sqrt{2})^{n+1}}{2} \end{bmatrix}. \end{aligned}$$

The first component of $\mathbf{x}_n$ is $x[n]$:

Closed form

$$x[n] = \frac{1}{2} \left[(1 + \sqrt{2})^{n+2} + (1 - \sqrt{2})^{n+2} \right], \quad n \geq 0.$$

As a check, this gives $x[0] = 3$, and the recursion gives $x[1] = 2 \cdot 3 + 1 = 7$, which the formula also gives.

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There is a quicker way to the same answer, based on one observation. Given the difference equation

$$x[n] = a_1 x[n-1] + a_2 x[n-2] + \cdots + a_k x[n-k],$$

try a solution of the form $x[n] = r^n$. It works exactly when

$$r^n = a_1 r^{n-1} + a_2 r^{n-2} + \cdots + a_k r^{n-k}$$

or, moving everything to one side,

$$r^n - a_1 r^{n-1} - a_2 r^{n-2} - \cdots - a_k r^{n-k} = 0 \quad \forall n.$$

Factoring out r^{n-k} leaves a polynomial in r :

$$\Rightarrow r^{n-k} \underbrace{(r^k - a_1 r^{k-1} - a_2 r^{k-2} - \cdots - a_k)}_{\text{characteristic equation of the difference equation}} = 0.$$

So r^n is a solution exactly when r is a root of the *characteristic equation* of the difference equation. This is the same polynomial (up to a sign) as $\det(A - rI)$ for the companion matrix A above, so the roots r are exactly the eigenvalues of

$$A = \begin{bmatrix} a_1 & \cdots & a_k \\ & & 0 \\ & I_{(k-1)} & \vdots \\ & & 0 \end{bmatrix}.$$

There are at most k distinct roots. If they are all distinct, a general solution is any combination of the corresponding powers:

General solution, distinct roots

$$x[n] = \alpha_1 r_1^n + \cdots + \alpha_k r_k^n = \sum_{i=1}^k \alpha_i r_i^n, \quad n \geq 0.$$

The constants α_i are fixed by the initial conditions, given either as $x[0], \dots, x[-k+1]$ or as $x[-1], \dots, x[-k]$. For the second form, evaluating the formula at $n = -1, \dots, -k$ gives a linear system for the α_i with an invertible coefficient matrix:

$$\underbrace{\begin{bmatrix} r_1^{-1} & r_2^{-1} & \cdots & r_k^{-1} \\ r_1^{-2} & r_2^{-2} & \cdots & r_k^{-2} \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{-k} & r_2^{-k} & \cdots & r_k^{-k} \end{bmatrix}}_{\text{invertible}} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{bmatrix} = \begin{bmatrix} x[-1] \\ \vdots \\ x[-k] \end{bmatrix}.$$

ex/ Apply this approach to the previous example.

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The difference equation is $x[n] = 2x[n-1] + x[n-2]$, with characteristic equation $r^2 - 2r - 1 = 0$ and roots $r_1 = 1 + \sqrt{2}$ and $r_2 = 1 - \sqrt{2}$. So $x[n] = \alpha_1 r_1^n + \alpha_2 r_2^n$. The initial conditions $x[0] = 3$ and $x[-1] = 1$ give two equations:

$$\begin{aligned} 3 &= \alpha_1 + \alpha_2 \\ 1 &= \underbrace{\frac{\alpha_1}{r_1} + \frac{\alpha_2}{r_2}}_{r_1 r_2 = r_2 \alpha_1 + r_1 \alpha_2} \Rightarrow \begin{aligned} \alpha_1 &= \frac{r_1 r_2 - 3r_1}{r_2 - r_1} = \frac{4 + 3\sqrt{2}}{2\sqrt{2}} \\ \alpha_2 &= 3 - \alpha_1 = \frac{3\sqrt{2} - 4}{2\sqrt{2}}. \end{aligned} \end{aligned}$$

Hence

$$x[n] = \left(\frac{4 + 3\sqrt{2}}{2\sqrt{2}} \right) (1 + \sqrt{2})^n + \left(\frac{3\sqrt{2} - 4}{2\sqrt{2}} \right) (1 - \sqrt{2})^n, \quad n \geq 0.$$

This looks different from the first answer, but it is the same sequence: both give $x[0] = 3$, $x[1] = 7$, $x[2] = 17$, $x[3] = 41, \dots$

ex/ The second approach also handles repeated roots, with one change. Consider

$$x[n] = 2x[n-1] - x[n-2], \quad x[-1] = 3, \quad x[-2] = 4.$$

The characteristic equation is $r^2 - 2r + 1 = 0$, so $r_1 = r_2 = 1$ is a double root. When a root is repeated, the solution takes the form

$$x[n] = \alpha_1 r_1^n + \alpha_2 n r_1^n.$$

(A proof for the similar situation of differential equations is given later.) The initial conditions give

$$\begin{aligned} x[-1] = 3 &= \alpha_1 r_1^{-1} - \alpha_2 r_1^{-1} = (\alpha_1 - \alpha_2) r_1^{-1} \\ x[-2] = 4 &= \alpha_1 r_1^{-2} - 2\alpha_2 r_1^{-2} = (\alpha_1 - 2\alpha_2) r_1^{-2}. \end{aligned}$$

With $r_1 = 1$ these are two simple equations:

$$\Rightarrow \left. \begin{aligned} \alpha_1 - \alpha_2 &= 3r_1 = 3 \\ \alpha_1 - 2\alpha_2 &= 4r_1^2 = 4 \end{aligned} \right\} \alpha_2 = -1, \quad \alpha_1 = 2.$$

Therefore

$$\Rightarrow x[n] = 2 - n, \quad n \geq 0.$$