

Section 1: Complex Numbers

MATH241 Lecture Notes, notes pages 1–7

Notation. Scalars are plain italic (a, α, z), vectors are bold ($\mathbf{u}, \mathbf{x}$), and matrices are italic capitals (A, B ; zero matrix O , identity I). The handwritten notes underline vectors once and matrices twice.

These notes are built on top of Orhan Arikan's linear algebra course notes.

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Introduction to Complex Numbers and Complex Algebra

★ (Chapter 21, Sections up to and including 21.3.2)

Complex numbers were introduced so that every polynomial equation has a complete set of solutions. The real numbers are not enough for this. The equation $x^2 + 1 = 0$ has no real solution at all, and the cubic below has only one real root, $x = 2$, because its quadratic factor $x^2 + x + 1$ has no real roots:

$$x^2 + 1 = 0 \quad \text{or} \quad x^3 - x^2 - x - 2 = (x - 2)(x^2 + x + 1) = 0$$

So the real numbers do not form a large enough set to contain the solutions of every polynomial equation. The complex numbers fix this.

Notation: A complex number is written $z \in \mathbb{C}$, in one of two equivalent ways:

$$\underbrace{z = a + ib}_{\text{mathematicians prefer this notation}} \quad \text{or} \quad \underbrace{z = a + jb}_{\text{electrical engineers prefer this notation}}$$

In both forms a and b are real numbers. We call a the *real part* of z and b its *imaginary part*, and we write

$$a \equiv \operatorname{Re}(a + jb), \quad b \equiv \operatorname{Im}(a + jb).$$

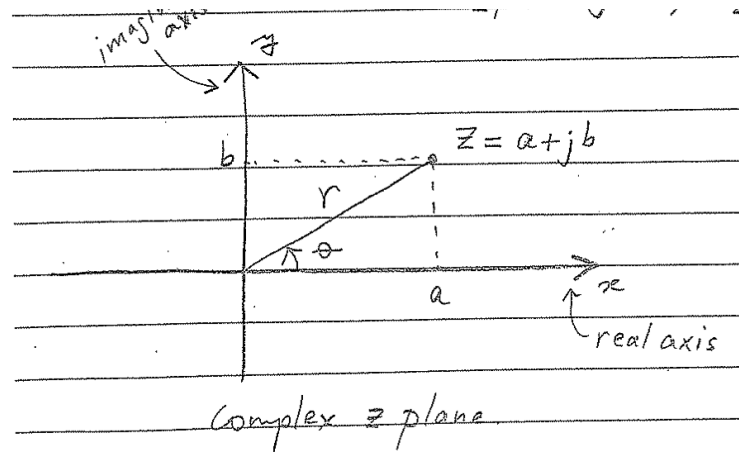
The symbol j stands for a root of $z^2 + 1 = 0$, so that $j^2 = -1$. These notes use j , as electrical engineers do.

Note that: the imaginary part of z is itself a real number, namely b and not jb . Also, every real number is a complex number with imaginary part zero, because $a \in \mathbb{R}$ gives $a + j0 \in \mathbb{C}$. In symbols, $\mathbb{R} \subset \mathbb{C}$.

Definitions:

- (i) A complex number is called *purely real* if $z = \operatorname{Re}(z)$, for example $z = 5 = 5 + j0$.
- (ii) A complex number is called *purely imaginary* if $z = j \operatorname{Im}(z)$, for example $z = j3 = 0 + j3$.
- (iii) Two complex numbers are equal, $z_1 = z_2$, if and only if their real parts are equal and their imaginary parts are equal: $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) = \operatorname{Im}(z_2)$. For example, $z_1 = 2 + j5$ and $z_2 = 2 + j5$ are equal.

A complex number $z = a + jb$ can be drawn as the point (a, b) in the plane. The horizontal axis is the real axis and the vertical axis is the imaginary axis, so z is a point in the *complex plane*.



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Algebraic Definitions

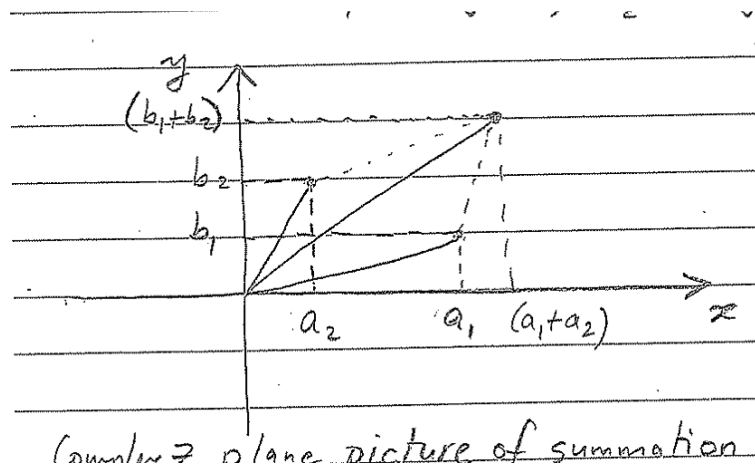
1. Summation Let $z_1 = a_1 + jb_1$ and $z_2 = a_2 + jb_2$. Their sum is the complex number $z = z_1 + z_2 = a + jb$, where the real parts add and the imaginary parts add separately:

Addition

$$(a_1 + jb_1) + (a_2 + jb_2) = (a_1 + a_2) + j(b_1 + b_2).$$

For example, with $z_1 = 2 + j3$ and $z_2 = 5 - j4$ we get $z_1 + z_2 = (2 + 5) + j(3 - 4) = 7 - j$.

In the complex plane, adding two numbers is the same as adding the two arrows from the origin: the sum is the far corner of the parallelogram built on z_1 and z_2 .



Properties of Complex Summation: For any $z_1, z_2 \in \mathbb{C}$:

Addition of complex numbers obeys the same rules as addition of real numbers.

- (i) $z_1 + z_2 = z_2 + z_1$ (commutative).
- (ii) $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ (associative).
- (iii) $z_1 + (0 + j0) = z_1$, so $0 + j0$ is the additive identity.

(iv) For every $z_1 \in \mathbb{C}$ there is a z_2 with $z_1 + z_2 = 0$.

proof Let $z_1 = a + jb$ and choose $z_2 = -a - jb$. Then

$$z_1 + z_2 = (a - a) + j(b - b) = 0 + j0 = 0.$$

(v) This additive inverse is unique.

proof Suppose $z_1 = a_1 + jb_1$ had two different additive inverses $z_2 = a_2 + jb_2$ and $z_3 = a_3 + jb_3$, that is,

$$z_1 + z_2 = 0, \quad z_1 + z_3 = 0 \quad \text{but} \quad z_2 \neq z_3.$$

Writing both sums out in terms of real and imaginary parts gives

$$\begin{aligned} z_1 + z_2 &= (a_1 + a_2) + j(b_1 + b_2), \\ z_1 + z_3 &= (a_1 + a_3) + j(b_1 + b_3). \end{aligned}$$

Since both sums equal 0, the real parts agree and the imaginary parts agree:

$$\left\{ \begin{array}{l} a_1 + a_2 = a_1 + a_3 \Rightarrow a_2 = a_3 \\ b_1 + b_2 = b_1 + b_3 \Rightarrow b_2 = b_3 \end{array} \right\} \Rightarrow z_2 = z_3.$$

This contradicts $z_2 \neq z_3$, so the inverse is unique. ■

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(vi) The unique additive inverse of $z = a + jb$ is written $-z = -a - jb$.

(vii) *Subtraction* is defined by adding the additive inverse:

$$z_1 - z_2 \equiv z_1 + (-z_2).$$

With $z_1 = a_1 + jb_1$ and $z_2 = a_2 + jb_2$ this gives

$$z_1 - z_2 = (a_1 + jb_1) + (-a_2 - jb_2) = (a_1 - a_2) + j(b_1 - b_2).$$

2. Multiplication Let $z_1 = a_1 + jb_1$ and $z_2 = a_2 + jb_2$. Their product $z = z_1 \cdot z_2$ is defined by

Multiplication

$$z = (a_1 + jb_1)(a_2 + jb_2) = (a_1a_2 - b_1b_2) + j(a_1b_2 + a_2b_1).$$

This is not an arbitrary rule. It is what you get by multiplying out the brackets as usual,

$$(a_1 + jb_1) \cdot (a_2 + jb_2) = a_1a_2 + ja_1b_2 + jb_1a_2 + j^2b_1b_2,$$

and then using $j^2 = -1$ to collect the real part $a_1a_2 - b_1b_2$ and the imaginary part $a_1b_2 + a_2b_1$.

example For $z_1 = 2 + j3$ and $z_2 = 5 - j4$:

$$z = z_1 \cdot z_2 = (2 \cdot 5 - 3 \cdot (-4)) + j(2 \cdot (-4) + 3 \cdot 5) = 22 + j7.$$

Properties of Complex Multiplication

Let $z_1 = a_1 + jb_1$, $z_2 = a_2 + jb_2$ and $z_3 = a_3 + jb_3$. Multiplication is commutative, associative and distributes over addition:

- (i) $z_1 z_2 = z_2 z_1$ (commutative).
- (ii) $z_1 (z_2 z_3) = (z_1 z_2) z_3$ (associative).
- (iii) $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$ (distributive).

Each of these follows by writing both sides out in terms of real and imaginary parts and comparing them; the calculation is routine.

- (iv) $1 \cdot z_1 = z_1$, so 1 is the multiplicative identity.
- (v) Every $z_1 \neq 0$ has a multiplicative inverse, a number z with $z_1 \cdot z = 1$.

proof Choose

$$z = \frac{a_1}{a_1^2 + b_1^2} - j \frac{b_1}{a_1^2 + b_1^2}.$$

Multiplying out with the product formula gives real part $\frac{a_1^2 + b_1^2}{a_1^2 + b_1^2} = 1$ and imaginary part $\frac{-a_1 b_1 + a_1 b_1}{a_1^2 + b_1^2} = 0$, so $z_1 \cdot z = 1$.

- (vi) This inverse is unique.

proof Suppose some $z_1 \neq 0$ had two different inverses z and $\hat{z}$.

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Multiply $z_1 z = 1$ by $\hat{z}$ and regroup:

$$\begin{aligned} z_1 \cdot z &= 1 \\ \Rightarrow \hat{z}(z_1 z) &= \hat{z} \cdot 1 \\ (\hat{z} z_1) z &= \hat{z} \\ \underbrace{1 \cdot z}_z &= \hat{z} \\ z &= \hat{z}. \end{aligned}$$

This contradicts the assumption that $z \neq \hat{z}$, so the inverse is unique. ■

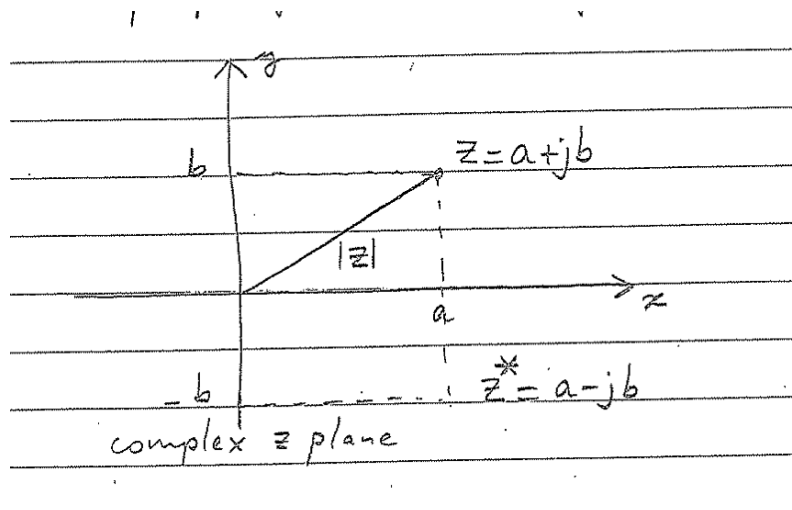
More very important Definitions

Two quantities are attached to every complex number. The *complex conjugate* of $z = a + jb$ flips the sign of the imaginary part, and the *modulus* (or magnitude) is the distance from the origin:

Conjugate and modulus

$$\bar{z} = a - jb = z^*, \quad |z| = \sqrt{a^2 + b^2} = \sqrt{z \cdot \bar{z}} = \sqrt{z \cdot z^*}.$$

Geometrically, z^* is the mirror image of z in the real axis, and $|z|$ is the length of the arrow from the origin to z .



For example, with $z = 2 + j3$ we have $z^* = 2 - j3$ and $|z| = \sqrt{2^2 + 3^2} = \sqrt{13}$.

Properties of complex conjugation:

Conjugation passes through sums and products:

- (i) $(z^*)^* = z$.
- (ii) $(z_1 + z_2)^* = z_1^* + z_2^*$.
- (iii) $(z_1 \cdot z_2)^* = z_1^* z_2^*$.

All three are checked by writing z_1 and z_2 in terms of their real and imaginary parts and comparing both sides.

Triangle Inequality

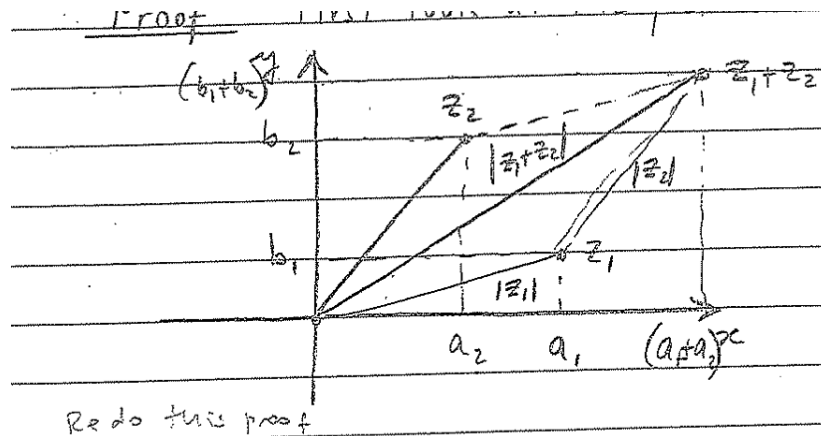
The next result says that going straight to $z_1 + z_2$ is never longer than going to z_1 first and then moving by z_2 .

Triangle inequality

$$\forall z_1 \text{ and } z_2, \quad |z_1 + z_2| \leq |z_1| + |z_2|.$$

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Proof The picture explains why the inequality is true. The three numbers $|z_1 + z_2|$, $|z_1|$ and $|z_2|$ are the lengths of the three sides of a triangle, and no side of a triangle is longer than the other two together.



To prove it algebraically, write $z_1 = a_1 + jb_1$ and $z_2 = a_2 + jb_2$, so that $z_1 + z_2 = (a_1 + a_2) + j(b_1 + b_2)$. Squaring the modulus and expanding gives

$$\begin{aligned} |z_1 + z_2|^2 &= (a_1 + a_2)^2 + (b_1 + b_2)^2 \\ &= (a_1^2 + b_1^2) + (a_2^2 + b_2^2) + 2(a_1a_2 + b_1b_2) \\ &\leq (a_1^2 + b_1^2) + (a_2^2 + b_2^2) + 2(|a_1||a_2| + |b_1||b_2|), \end{aligned}$$

where the last step uses $xy \leq |x||y|$. The first two brackets are $|z_1|^2$ and $|z_2|^2$. To handle the last term we claim that

$$|a_1||a_2| + |b_1||b_2| \leq \sqrt{a_1^2 + b_1^2} \sqrt{a_2^2 + b_2^2} = |z_1||z_2|.$$

Both sides are non-negative, so it is enough to compare their squares:

$$(|a_1||a_2| + |b_1||b_2|)^2 \leq (a_1^2 + b_1^2)(a_2^2 + b_2^2).$$

Expanding both sides and cancelling the common terms $a_1^2a_2^2 + b_1^2b_2^2$, this is equivalent to

$$2|a_1||a_2||b_1||b_2| \leq a_1^2b_2^2 + b_1^2a_2^2 \iff 0 \leq (|a_1||b_2| - |b_1||a_2|)^2,$$

which is true because a square is never negative. Putting the pieces together,

$$\begin{aligned} |z_1 + z_2|^2 &\leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2| = (|z_1| + |z_2|)^2 \\ \Rightarrow |z_1 + z_2| &\leq |z_1| + |z_2|. \quad \blacksquare \end{aligned}$$

Def'n: Division is defined through the multiplicative inverse. If $z_2 \neq 0$,

Division

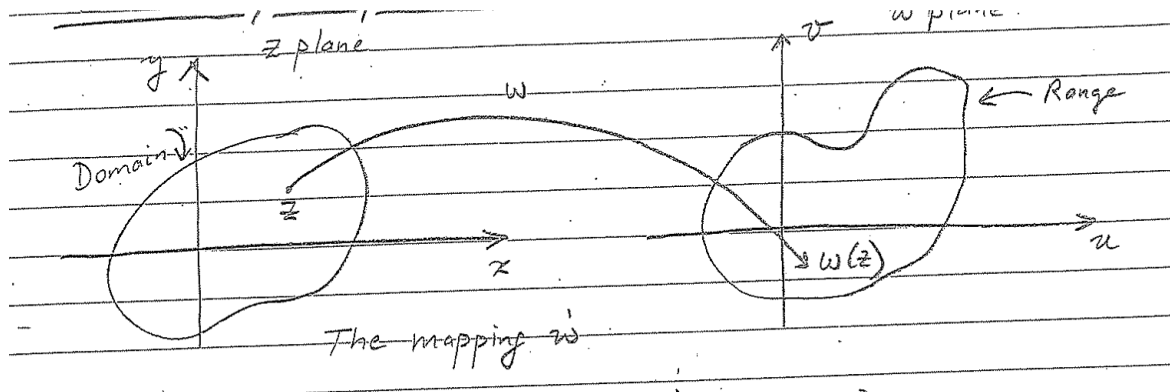
$$\frac{z_1}{z_2} = z_1 \cdot z, \quad \text{where } z \text{ is the unique multiplicative inverse of } z_2.$$

In practice we multiply the top and the bottom by the conjugate of the denominator, which makes the denominator real: ex/ Compute $\frac{2+j3}{3-j4}$.

$$\begin{aligned} \frac{2+j3}{3-j4} &= (2+j3) \cdot \frac{(3+j4)}{(3-j4)(3+j4)} = \frac{(2+j3)(3+j4)}{25} \\ &= \frac{(6-12) + j(8+9)}{25} = -\frac{6}{25} + j\frac{17}{25}. \end{aligned}$$

Elementary Complex Functions

A complex function w takes a complex number z in its *domain* and returns a complex number $w(z)$ in its *range*. Because both z and $w(z)$ live in planes, we picture a function as a map from one plane to another.



If $z = x + jy$, the output has a real part and an imaginary part, each depending on x and y :

$$w(z) = w(x + jy) = u(x, y) + jv(x, y).$$

ex/ *Squaring*. Let $w(z) = z^2$ with the first quadrant of the z plane as its domain, that is, $0 < x < \infty$ and $0 < y < \infty$. Then

$$w(x + jy) = (x + jy)(x + jy) = x^2 - y^2 + j2xy,$$

so $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. Since x and y are both positive, $v > 0$, while u can be any real number. The range is therefore $-\infty < u < \infty$ and $0 < v < \infty$, the upper half plane.

ex/ *Conjugation*. Let $w(z) = z^*$ on the entire complex plane. Then $w(x + jy) = x - jy$, so $u(x, y) = x$ and $v(x, y) = -y$. This reflects the plane in the real axis, and the range is the whole complex plane.

ex/ *The exponential function*. The complex exponential is defined by the same power series as the real one:

$$w(z) = e^z \equiv 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots$$

For $z = x + jy$ it splits into a real exponential and a pure phase, $e^{x+jy} = e^x \cdot e^{jy}$ (we use this fact without proof here). To understand e^{jy} , put jy into the series. The powers of j repeat, $j^2 = -1$, $j^3 = -j$, $j^4 = 1$, so the terms separate into a real part with the even powers and an imaginary part with the odd powers:

$$\begin{aligned} e^{jy} &= 1 + jy + \frac{(jy)^2}{2!} + \cdots + \frac{(jy)^n}{n!} + \cdots \\ &= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots + \frac{(-1)^k y^{2k}}{(2k)!} + \cdots \right) \\ &\quad + j \left(y - \frac{y^3}{3!} + \cdots + \frac{(-1)^k y^{2k+1}}{(2k+1)!} + \cdots \right). \end{aligned}$$

The first bracket is the series of $\cos y$ and the second is the series of $\sin y$. This gives one of the most useful formulas in engineering mathematics:

Euler's formula

$$e^{jy} = \cos y + j \sin y.$$

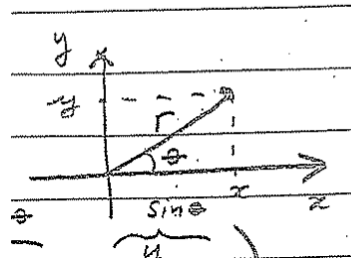
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Putting Euler's formula back into $e^z = e^x \cdot e^{jy}$ gives the real and imaginary parts of the exponential:

$$\begin{aligned} w(z) = e^z &= e^x \cdot e^{jy} = e^x (\cos y + j \sin y) \\ &= e^x \cos y + j e^x \sin y \\ &= u(x, y) + jv(x, y), \end{aligned}$$

with $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$. In particular $|e^z| = e^x > 0$, so e^z is never zero, for any complex z .

Polar representation A point in the plane can be described by its distance $r = |z|$ from the origin and its angle θ from the real axis, instead of by x and y .



Factoring out the length $r = \sqrt{x^2 + y^2}$ leaves a bracket whose two entries are $\cos \theta$ and $\sin \theta$, and by Euler's formula the bracket is $e^{j\theta}$:

$$z = x + jy = \sqrt{x^2 + y^2} \underbrace{\left(\underbrace{\frac{x}{\sqrt{x^2 + y^2}}}_{\cos \theta} + j \underbrace{\frac{y}{\sqrt{x^2 + y^2}}}_{\sin \theta} \right)}_{e^{j\theta}}$$

Polar form

$$z = r e^{j\theta}, \quad r = |z| \text{ (magnitude), } \theta \text{ (phase).}$$